Luxury Brand Licensing: Free Money or Brand Dilution?

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Licensing enables luxury brands to reach out to their aspirational, low-end consumers ('followers') who value a brand more when more high-end consumers ('snobs') use it. However, over-licensing might dilute the brand for snobs who value brand exclusivity. We develop a game-theoretic model to study these two countervailing forces of licensing. When a brand charges its licensee an upfront fixed fee, we find that the monopolist brand should not license when snobs' desire for exclusivity is high. However, in a duopoly, fixed-fee contracts *soften* price competition so that licensing (even for free) is *always* profitable for both brands. Interestingly, in equilibrium, competing brands may still prefer not licensing even though both would be better off if they could commit to licensing. We also analyze a duopoly where brands charge their licensees a royalty fee per unit sold and find that brands may become worse off under royalty contracts because, relative to fixed-fee contracts, royalty contracts *intensify* price competition. Finally, we compare a decentralized system with fixed-fee licensing and a centralized system with umbrella branding. We find that umbrella branding *intensifies* price competition and fixed-fee contracts are more profitable when followers' aspiration is strong, which implies that a decentralized licensing system can be more efficient.

Key words: luxury brand; licensing; conspicuous goods; competition; fixed fee; royalty

1. Introduction

A brand is a name, term, sign, symbol or design that contributes to the value of a product beyond its functional use [Farquhar, 1989]. A great example is Louis Vuitton: a luxury brand that has $33.6 billion in brand value [Forbes, 2018]. Luxury brands usually build their initial brand image/reputation by producing and selling unsurpassed quality products in certain categories for a niche customer segment. For example, Giorgio Armani offers a high-end designer clothing line for men and women, Gucci designs and manufactures handbags from fine leather, and Bang & Olufsen makes high-end, uniquely designed electronics for discerning customers.

Given their strong brand image, luxury brands become a platform for launching new products for aspirational consumers. Specifically, once the market for a luxury brand's primary (high-quality) products matures, it faces pressure from investors to grow and capture aspirational consumers by leveraging its brand image to expand its presence quickly in new product categories via licensing. Specifically, many luxury brands license their brand names to firms (licensees) with the expertise to design, produce and sell licensed products. For example, Burberry, Gucci and Hugo Boss license their fragrance and/or cosmetics business to Coty – one of world's largest beauty and fragrance companies [Sandle, 2017]. In the same vein, Bulgari, Ferragamo, Prada and Versace license their eyewear to Luxottica – the world's largest eyewear company [License Global, 2004]. In 2017, retail sales of licensed goods reached $271.6 billion, and the bulk of this sales figure was generated from the sales of licensed goods that bear different luxury brand names [Greene, 2009; Licensing Industry Merchandisers' Association, 2017]. In general, licensed products are of lower quality and are significantly more affordable products than the primary products [License Global, 2004; Amaldoss and Jain, 2015]. For example, many consumers cannot afford Gucci handbags, but they can show their aspiration by purchasing licensed products such as Gucci fragrance [Centre for Fashion Enterprise, 2012]. Therefore, licensing creates an opportunity for luxury brands (especially, small luxury brands with limited resources) to build a presence in the market of aspirational consumers and to venture into new product categories with greater ease than if they were to produce in-house.

On the other hand, licensing (if not carefully managed) might come at a price and tarnish the image of a luxury brand, especially if the brand over-extends and licenses its brand name to too many products. When making their purchasing decisions, consumers of luxury brands' primary products are exclusivity-seeking (snobbish) and also consider the composition (i.e., type and number) of consumers adopting the brand [Amaldoss and Jain, 2008, 2015]. Due to these social effects, the luxury brand becomes less exclusive and its image is diluted if it over-extends the brand by licensing too much. This is the exact reason why Gucci, Yves Saint Laurent (YSL) and Burberry failed when they first attempted to license in the 1980s and 1990s [License Global, 2004].[1] Consequently, there are two countervailing forces of licensing: it creates an opportunity for brands to grow by reaching out to their aspirational consumers (i.e., followers) who care about the brand popularity; however, it also reduces brands' attractiveness for consumers purchasing brands' primary products (i.e., snobs) who care about brands' exclusivity, and so may dilute their image. These two countervailing forces motivate us to develop a game-theoretic model to examine how these interactions between snobs and followers affect a luxury brand's licensing strategy.

In our parsimonious model, two competing *brands* produce and sell high-quality products in the same category (e.g., handbags). At the same time, both brands consider licensing their brand names to two competing *licensees* which have expertise in producing and selling low-quality products in a different category (e.g., eyewear). In the base model, each brand uses a fixed-fee licensing contract to license its name to the corresponding licensee. Under a fixed-fee contract, the licensee pays the brand a lump-sum fixed fee upfront so that the licensee has the right to produce and sell certain products that carry the brand name for an extended time frame [Centre for Fashion Enterprise, 2012; Chevalier and Mazzalovo, 2012]. On the consumer side, we consider two segments, namely, snobs and followers. Snobs value exclusivity and they do not want

[1] In the 1980s, Gucci licensed its brand name to different licensees who produced over 22,000 products such as alcohol, key chains and even toilet paper and distributed them through department stores. This licensing strategy backfired because the Gucci brand was diluted and its image was associated with 'drug stores' [Jackson et al., 2002]. Gucci gradually recovered its image by limiting the number of its licenses and by having tighter controls over its licensees.

to be associated with followers (i.e., *direct negative popularity effect*). However, followers have a strong desire to assimilate the same brand adopted by snobs so that they value licensees' product more as more snobs purchase the brand (i.e., *direct positive popularity effect*). Snobs also have a higher willingness to pay compared to the followers. Therefore, brands offer their high-quality products to snobs, while licensees offer their lower-quality products to followers. Using this modeling framework, we examine four research questions:

1. In a monopolistic environment, what are the favourable conditions for a luxury brand to license its brand name to a licensee?
2. In a duopolistic environment, how would competition between brands and the strategic interplay between snobs and followers affect brands' equilibrium licensing strategies?
3. How would our results change as the licensing contract between a brand and its licensee is changed from a fixed-fee contract to a royalty contract where the licensee pays a per-unit royalty fee for each unit sold?
4. Why do some luxury brands license a lot while others do not license at all and instead prefer to produce in-house and use umbrella branding?

Our analysis yields the following results. We show that the monopolist should not license its brand name when the snobs' negative popularity effect is high. However, it is interesting to note that this result does not hold in a duopoly. Specifically, under competition in a duopolistic environment, we find that licensing is always beneficial for both brands, even when the brands license their brand names for free. This interesting result is driven by *the indirect (strategic) effect* of licensing that softens price competition between brands so that both brands can afford to increase their selling price without losing market share. To see this, suppose that one of the brands increases the price of its high-quality product. In doing so, this brand will lose some market share in the snobs' market, which will lower the appeal of the licensed products in the followers' market. As fewer followers purchase the licensed product, the brand's high-quality product will become more attractive to the snobs. Hence, the brand can afford to increase its price without affecting its market share significantly in the snobs' market.

By comparing each brand's payoff under different licensing strategies, we characterize brands' equilibrium licensing strategies under fixed-fee contracts and thereby identify the impact of market structure and conspicuous consumption on brands' licensing decisions. First, when the snobs' negative popularity effect is above a certain threshold, each brand would have earned more if they could both commit to licensing. However, in the absence of such a commitment, both brands face a prisoner's dilemma and do not license in equilibrium. Second, when the snobs' negative popularity effect is below a certain threshold, we map out the conditions under which both brands, or only one brand, would license in equilibrium. Interestingly, we find that symmetric brands can prefer asymmetric licensing strategies due to conspicuous consumption.

Then, we extend our analysis to the case when the licensing fee is royalty based and the licensee pays the brand a per-unit royalty fee for each unit sold [License Global, 2004; Greene, 2009; Centre for Fashion Enterprise, 2012]. We find that the royalty fee provides an additional leverage for the brand to control its licensee's price (and demand). Consequently, it can enable the brand to manage the impact of the negative popularity effect on the snobs' demand. Due to this extra edge, we find that a royalty licensing contract is preferred over a fixed-fee contract in a wider range of the snobs' negative popularity effect in the monopolistic setting. In a duopolistic environment, we find that a royalty licensing contract can create a new *royalty effect* that intensifies the competition between brands so that both brands will lower their prices when they both license. This result is driven by the fact that, under a royalty licensing contract, both brands can earn more royalties by increasing the followers' demand for the licensed product. Because of the followers' positive popularity effect, both brands can increase the followers' demand by increasing the snobs' demand. As both brands compete for higher demand in the snob market, the price competition between brands is intensified. As a result, when both brands license, their prices are lower under a royalty contract than under a fixed-fee contract. When the followers' positive popularity effect is sufficiently high, the competition between brands in the snob market is very intense and the royalty contract is dominated by the fixed-fee contract. We also characterize licensing strategies of brands in equilibrium under the royalty contract, identify cases where two brands use symmetric or asymmetric licensing strategies under royalty contracts, and show that the equilibrium licensing strategies are similar to those under fixed-fee contacts.

Finally, we examine the umbrella branding strategy under which brands have a centralized control for producing low-quality products in-house. A key belief in the operations and supply chain literature on supply contracts is that decentralization can introduce inefficiencies (e.g., double marginalization, etc.) and therefore the centralized systems are more efficient than decentralized systems [Pasternack, 1985; Taylor, 2002; Cachon and Lariviere, 2005]). In contrast to this key belief, we find that under competition, when followers' positive popularity effect is sufficiently high, umbrella branding intensifies price competition between brands significantly, and umbrella branding is dominated by a fixed-fee licensing contract.

This paper is organized as follows. In the following section, we review the related literature. In §3, we present our model and assumptions. We study the fixed-fee licensing contract in the monopoly setting in §4 and in the duopoly setting in §5. We discuss our duopoly results under a royalty licensing contract and the umbrella branding strategy in §6 and §7, respectively. Finally, §8 concludes the paper. All proofs are provided in Appendix E.

2. Literature Review

Our paper is related to two research streams: *licensing* and *conspicuous consumption*. First, the economics literature on patent licensing dates back to Arrow [1962]. Kamien and Schwartz [1982] extend Arrow's work by considering royalty licensing contracts in a Cournot oligopoly. Using different game-theoretic frameworks, several economists analyzed different licensing strategies of an inventor (licensor). Kamien and Tauman [1986], Katz and Shapiro [1986], Kamien [1992], and Kamien et al. [1992] show that, when there is perfect information, fixed-fee licensing outperforms royalty licensing for the inventor when the inventor (licensor) is an outsider and does not compete with its licensees. However, royalty licensing dominates when the inventor is an insider and competes with its licensees, and/or when there is demand/cost uncertainty or information asymmetry; see Bousquet et al. [1998], Beggs [1992], Gallini and Wright [1990], Wang [1998], and Choi [2001]. Unlike the economic literature on patent licensing, we examine the issue of brand licensing of luxury goods by capturing the conspicuous consumer behavior. The literature on brand licensing is limited. Marjit et al. [2007] consider a Cournot duopoly in which a superior foreign brand and a less-reputed domestic brand compete in the domestic market and examine the conditions under which the foreign brand should license through a fixed-fee contract. Basak and Mukherjee [2014] extend Marjit et al. [2007] and show that foreign brands should always license by offering a fixed-fee plus per-unit royalty contract. In contrast to Marjit et al. [2007] and Basak and Mukherjee [2014], we focus on the licensing of conspicuous goods by incorporating the snobs' negative popularity effect and the followers' positive popularity effect.

Second, the literature on conspicuous consumption dates back to Veblen [1899] who postulates that individuals consume conspicuous products to signal their wealth and social status. Leibenstein [1950] suggests that the price and quantity consumed by others affect the value of a product for some consumers. Specifically, Becker [1991], and Corneo and Jeanne [1997] show that the demand for a product may increase in its price when consumers are followers (conformists) and value a product more when more people purchase it. Amaldoss and Jain [2005a] develop a model of conspicuous consumption and analyze how demand and price of a monopolist firm are affected by snobs and followers. Amaldoss and Jain [2005b] extend Amaldoss and Jain [2005a] to a duopoly setting and show that the snobs' negative popularity effect leads to higher price and profits while the followers' positive popularity effect has the opposite impact. Rao and Schaefer [2013] model the impact of quality and status-related consumer considerations on firms' pricing and product management decisions. They find that a product's high intrinsic quality leads to exclusivity and results in higher social utility for consumers who value status. Arifoğlu et al. [2019] consider snobbish consumers with heterogenous (high and low) valuations. They find that snobbish consumer behavior leads to buying frenzies and price markdowns. Unlike these papers, there are also several papers that study the impact of conspicuous consumption on pricing and product management strategies of firms selling multiple products. In a setting where two firms with different qualities compete, Balachander and Stock [2009] show that offering 'limited editions' intensifies the competition and benefits only the high-quality firm. When there are snobs and followers in the market, Amaldoss and Jain [2008] show that the snobs' negative popularity effect can encourage a monopoly firm to offer limited editions. Unlike these papers, we focus on firms that sell multiple products and analyze their brand extension strategies via licensing (as our main focus) and umbrella branding.

In this paper we adopt the modeling framework developed by Amaldoss and Jain [2015] that captures: (1) the snobs' negative popularity effect; and (2) the followers' positive popularity effect. However, our paper is fundamentally different from Amaldoss and Jain [2015] in four aspects.

- First, we examine the issue of brand licensing. In our model, each brand (licensor) licenses its brand name to a licensee that produces and sells a different low-quality product to followers. We deal with a 'decentralized' system because the brand does not have direct control over its licensee's pricing decision. Hence, our model and our analysis are completely different from Amaldoss and Jain [2015] who focus on a different issue (i.e., umbrella branding) in a different setting (i.e., 'centralized' system in which the brand has complete control on the pricing of its umbrella brands).
- Second, we examine different licensing contracts (fixed-fee and royalty-based contract) arising from a decentralized system. This kind of contracting issue does not exist in a centralized system as examined in Amaldoss and Jain [2015]. Hence, our paper complements Amaldoss and Jain [2015].
- Third, we obtain some new findings. We find that fixed-fee licensing contracts soften price competition between brands, and both brands become better off even when they license for free. In addition, we examine how conspicuous consumer behavior affects the licensing strategy to be adopted by two competing firms in equilibrium.
- Fourth, we obtain new managerial insights. When we compare decentralized licensing and centralized umbrella branding, we find that fixed-fee licensing dominates umbrella branding, especially when the followers' positive popularity effect is strong. This result is due to the fact that competition is softened under licensing and yet competition is intensified under umbrella branding.

Overall, our paper is the first to examine different licensing strategies in the context of conspicuous consumption under competition.

3. Model Preliminaries

Consider two competing luxury brands A and B that produce and sell the same category of 'high-quality' and more expensive product(s) (e.g., Fendi and Gucci for leather goods). To grow its revenue quickly, each brand considers licensing its brand name to its corresponding licensee (say, licensee a for brand A and licensee b for brand B) who has expertise in designing, producing and selling 'low-quality' and cheaper products in a different category (e.g., cologne) that carries the corresponding brand name. In our model, we assume that the unit production cost of high-quality goods (produced by the brands) is equal to c, which is higher than the unit production cost of low-quality goods (produced by licensees) that we normalize to 0.

Market structure. The high-quality goods of both brands (A and B) are sold in market s comprised of 'snobs' with market size equal to 1. In a similar spirit, the low-quality goods produced by the licensees (a and b) are sold in a different market f comprised of 'followers' with market size β. (We impose no assumption on β; however, licensed products are often sold in mass markets so that $\beta > 1$). In our model, we assume that the markets s and f are 'separate' in the sense that snobs will never purchase the licensed goods that are perceived to be low quality, and followers will never purchase the luxury goods that are too expensive [Centre for Fashion Enterprise, 2012; Amaldoss and Jain, 2015]. In doing so, we can isolate the 'competition effect' within each market and the 'popularity effect' across markets that can be described as follows.

Intra-market *Competition: Functional Effect.* For both snobs and followers, a product's value is influenced by *functional* and *social* effects. Within each market s (or f), we use the Hotelling model to capture heterogeneous preferences for the functionality of the product so that all snobs are uniformly distributed over the line $[0,1]$, where brand A's product is located at 0 and B at 1. Hence, for a snob who is located at θ, his/her functional value for brand A's product is $(v_s - t_s\theta)$ and for brand B's product is $(v_s - t_s(1-\theta))$ so that both firms engage in price competition within market s. Here, v_s is the base valuation of the product associated with each brand and t_s represents the 'fit-cost-loss' coefficient.

Using a similar construct, we assume that licensed product a is located at 0 and b at 1, a follower located at θ values product a at $(v_f - t_f\theta)$ and values b at $(v_f - t_f(1-\theta))$ so that licensees a and b engage in price competition within market f. To capture the notion that the luxury brand carries a higher valuation than its licensed product, we assume that $v_s > v_f$.

Inter-market *Popularity: Social Effect.* Through licensing, a brand's name is exposed to both snobs and followers in markets s and f, which can bring about 'positive' and 'negative' popularity effects among snobs and followers. First, snobs despise the popularity of licensed products sold in market f so that a snob's utility derived from purchasing brand I is decreasing in $D_f^{i(e)}$; i.e., his/her expectation about the proportion of followers purchasing the licensed product i in market f. Accounting for its functional value, the net utility that a snob located at θ will derive from purchasing product A is given by:

$$U_s^A(\theta) = v_s - t_s\theta - \lambda_s \beta D_f^{a(e)} - p^A, \tag{1}$$

where λ_s represents the snobs' 'negative popularity effect' of licensing a brand in market f, and p^A is the selling price. The net utility for purchasing brand B can be obtained in the same manner.

Second, followers in market f interpret the popularity of a brand in market s as a form of endorsement; hence, a follower's utility derived from purchasing licensed product i is 'increasing' in $D_s^{I(e)}$; i.e., his/her expectation about the proportion of snobs purchasing the product of luxury brand I in market s. More formally, the net utility that a follower located at θ will derive from purchasing the licensed product a is given by:

$$U_f^a(\theta) = v_f - t_f\theta + \lambda_f D_s^{A(e)} - p^a, \tag{2}$$

where λ_f represents the followers' 'positive popularity effect' associated with the sales of the luxury brand in market s, and p^a is the selling price. We can obtain a similar expression for the net utility associated with licensee b's product.

Profit Function of a Brand and its Licensee. Each brand I ($I = A, B$) licenses its name through a contract which specifies a transfer payment T^I to be collected from its corresponding licensee i ($i = a, b$). To explicate our analysis, we first consider fixed-fee contracts that are commonly used in the luxury goods industry (or by upcoming designers) to generate capital or sustain their growth [Centre for Fashion Enterprise, 2012; Chevalier and Mazzalovo, 2012]. Under a fixed-fee contract, the licensee i pays a fixed fee k^I (lump-sum payment) to luxury brand I, i.e., $T^I = k^I$. (In a later section, we shall consider royalty contracts under which the transfer payment is based on the demand of the licensed product (i.e., βD_f^i) so that $T^I = r^I \beta D_f^i$, where r^I is the royalty fee per unit sold in market f.)

Let D_s^I ($I = A, B$) be the *actual* proportion of snobs purchasing from luxury brand I and D_f^i ($i = a, b$) be the *actual* proportion of followers purchasing from licensee i. By accounting for the transfer payment T^I associated with a licensing contract (fixed-fee or royalty contract), the profits of brand I and its corresponding licensee i can be written as:

$$\Pi^I\left(p^I\right) = (p^I - c)D_s^I + T^I, \tag{3}$$

$$\Pi^i\left(p^i\right) = p^i \beta D_f^i - T^I. \tag{4}$$

Rational Expectations Equilibrium. We note that snobs' or followers' expectations, i.e., $D_f^{i(e)}$ ($i = a, b$) and $D_s^{I(e)}$ ($I = A, B$), can be different from the actual consumption, i.e., D_f^i ($i = a, b$) and D_s^I ($I = A, B$). However, by using the concept of rational expectations equilibrium [Amaldoss and Jain, 2005a; Su and Zhang, 2008; Liu and van Ryzin, 2008; Amaldoss and Jain, 2015], the actual proportion is equal to the anticipated proportion in equilibrium so that $D_s^I = D_s^{I(e)}$ for $I = A, B$ and $D_f^i = D_f^{i(e)}$ for $i = a, b$.

4. Monopoly: Fixed-Fee Contracts

Consider the case when brand A operates as a monopoly located at 0 in market s. Brand A may license its brand name to licensee a located at 0 in market f. For ease of exposition, we shall restrict our analysis to the case when market s is fully covered (i.e., the resulting $D_s^A = 1$), which happens when the base valuation v_s is sufficiently high. (In Appendix A, we show that our results in the monopoly setting continue to hold even when the market s is not fully covered.)

We consider the following sequence of events. First, brand A decides whether to license or not, and if it licenses, it determines and offers a fixed-fee contract ($T^A = k^A$) to licensee a. If licensee a agrees to pay the fixed fee k^A, then brand A sets its price p^A for its luxury goods to be sold in market s and licensee a sets its own price p^a for its licensed product to be sold in market f simultaneously. (If licensee a rejects the contract, then no licensing will occur and brand A operates as a monopoly in market s.) Lastly, snobs in market s decide whether to purchase brand A's product and followers in market f decide whether to purchase licensee a's licensed product. This sequence of events is modeled as a sequential game.

4.1. No Licensing (NL)

As a benchmark, suppose brand A does not license its name to licensee a so that $D_f^a = 0$ and $T^A = 0$. To ensure that the entire market s is covered so that the snob located at $\theta = 1$ will purchase the product in this case, it is optimal for brand A to set its price equal to $p^A = v_s - t_s$ so that $U^A(1) = 0$. Substituting the optimal price $p^A = v_s - t_s$ into brand A's profit function in (3) along with the fact that market s is fully covered ($D_s^A = 1$) and there is no license fee (i.e., $T^A = 0$), brand A's equilibrium profit for the case of no licensing is given by:

$$\Pi^A(NL) = v_s - t_s - c. \tag{5}$$

4.2. Licensing under a Fixed-Fee Contract (F)

We now consider the case when brand A licenses its brand name to licensee a by charging a fixed-fee $T^A = k^A$. We use backward induction to characterize the equilibrium of the sequential game as described above. To begin, let us consider the consumer's problem. If licensee a sells its licensed good in market f, then a follower located at θ will purchase if his/her net utility $U_f^a(\theta) = v_f - t_f\theta + \lambda_f D_s^{A(e)} - p^a \geq 0$. As we restrict our analysis for the case when brand A will set its price to ensure that the entire market s is fully covered, followers in the market f anticipates that, i.e., $D_s^{A(e)} = 1$. In this case, by (2), the marginal follower θ_f who is indifferent between purchasing and not purchasing licensee a's product is given by:

$$\theta_f = \min\left\{\frac{v_f + \lambda_f - p^a}{t_f}, 1\right\}. \tag{6}$$

Similarly, by rational expectations, snobs in market s can anticipate that $D_f^{a(e)} = \theta_f$ so that the demand for licensee a's product is equal to $\beta\theta_f$. Combine this observation with the fact that market s is fully covered, the snob located at $\theta = 1$ will purchase luxury brand A's product, if his/her net utility $U_s^A(1) = v_s - t_s - \lambda_s\beta\theta_f - p^A \geq 0$. This implies that it is optimal for brand A to set its price $p^A = v_s - t_s - \lambda_s\beta\theta_f$.

Next, we consider luxury brand A's and licensee a's problems. Given that $D_f^a = \theta_f$ as stated in (6) and $D_s^{A(e)} = 1$, licensee a will determine its optimal selling price that maximizes its profit as stated in (4) for any given fixed fee k^A. By considering the first-order condition along with the bound on p^a, we get:

$$p^a = \begin{cases} \frac{v_f + \lambda_f}{2}, & \text{if } \lambda_f < 2t_f - v_f, \\ v_f + \lambda_f - t_f, & \text{if } \lambda_f \geq 2t_f - v_f. \end{cases}$$

By substituting p^a into (6), we can retrieve θ_f and then the corresponding brand A's optimal price $p^A = v_s - t_s - \lambda_s\beta\theta_f$, which can be rewritten as:

$$p^A = \begin{cases} v_s - t_s - \lambda_s\beta\frac{v_f + \lambda_f}{2t_f}, & \text{if } \lambda_f < 2t_f - v_f, \\ v_s - t_s - \lambda_s\beta, & \text{if } \lambda_f \geq 2t_f - v_f. \end{cases} \tag{7}$$

A direct comparison of the selling prices stated in (7) and in the case of no licensing reveals that licensing a luxury brand will cause the brand to lower its selling price; i.e., $p^A < v_s - t_s$. This phenomenon is caused by the snobs' *negative popularity effect* λ_s: as licensee a sells its low-quality product in market f, the 'effective' valuation of brand A's luxury goods in market s decreases due to the snobs' negative attitude (i.e.,λ_s) towards the popularity of the licensed product in market f. As the valuation drops, brand A has to lower its selling price to ensure that market s continues to be fully covered. While brand A suffers from a lower margin for its luxury goods, it recovers this loss from the licensing fee k^A to be collected from licensee a.

Next, we determine the equilibrium fixed fee k^A. By substituting p^a into (6), we can retreive θ_f and determine licensee a's profit given in (4) for any given k^A. In this case, it is optimal for brand A to set the fixed lump-sum payment k^A to extract the entire surplus so that the licensee a ends up with zero profit (and yet licensee a will accept the licensing contract). In this case, it can be shown that k^A satisfies:

$$k^A = \begin{cases} \beta\frac{(v_f+\lambda_f)^2}{4t_f}, & \text{if } \lambda_f < 2t_f - v_f, \\ \beta(v_f - t_f + \lambda_f), & \text{if } \lambda_f \geq 2t_f - v_f. \end{cases}$$

Notice that the equilibrium lump-sum payment k^A is strictly increasing in λ_f. Hence, as followers appreciate popularity more (i.e., as λ_f increases), brand A can command a higher fixed licensing fee k^A.

Using (3), through substitution along with the fact that $D_s^A = 1$ (full coverage of market s), it is easy to show that brand A's profit in equilibrium can be written as:

$$\Pi^A(F) = \begin{cases} v_s - t_s - c + \beta\frac{v_f+\lambda_f}{2t_f}\left(\frac{v_f+\lambda_f}{2} - \lambda_s\right), & \text{if } \lambda_f < 2t_f - v_f, \\ v_s - t_s - c + \beta\left(v_f + \lambda_f - \lambda_s - t_f\right), & \text{if } \lambda_f \geq 2t_f - v_f. \end{cases} \tag{8}$$

Comparing brand A's profit with fixed-fee contract $\Pi^A(F)$ and its profit without licensing $\Pi^A(NL) = v_s - t_s - c$, we get Proposition 1.

PROPOSITION 1. *Suppose market s is fully covered. Brand A licenses its brand name to licensee a if and only if: (1) $\lambda_s < \frac{v_f+\lambda_f}{2}$ when $\lambda_f < 2t_f - v_f$; or (2) $\lambda_s < v_f + \lambda_f - t_f$ when $\lambda_f \geq 2t_f - v_f$.*

Proposition 1 implies that the licensing decision of a monopoly is driven by the snobs' negative popularity effect λ_s. Specifically, when the snobs are less sensitive towards the popularity of the brand in market f, i.e., when λ_s is sufficiently lower than the followers' positive popularity effect λ_f, brand A can afford to license its name to licensee a because the gain through the fixed fee k^A outweighs the loss caused by the lower profit margin in market s (i.e., lower selling price p^A).

In summary, when brand A operates in market s as a monopoly, we find that brand A should license its brand to licensee a if, and only if, λ_s is sufficiently lower than λ_f. Will this result hold when two luxury brands compete in market s and two licensees compete in market f? We examine this question next.

5. Duopoly: Fixed-Fee Contracts

We now consider the case when both brands A and B sell their luxury goods in market s and both consider licensing through fixed-fee contracts to their respective licensees a and b who sell licensed products in market f. For tractability, we assume that both markets s and f are fully covered so that $D_s^A + D_s^B = 1$, and $D_f^a + D_f^b = 1$.[2] We also assume that competing with the other brand in market s and/or licensing its brand name in market f always makes a brand worse off compared to the case where it is a monopoly in market s. This assumption and our assumption that market s is fully covered require that snobs' base valuation v_s is sufficiently high enough. We consider a similar sequence of events as described in the last section for the monopoly case. Because brand A and brand B are symmetric, it suffices to consider three cases: (i) both brands do not license, (ii) both brands license, and (iii) only one brand licenses. Then, by comparing the equilibrium payoffs associated with these three cases for brands A and B, we characterize the equilibrium licensing strategies of both brands under fixed-fee contracts.

5.1. Both brands do not license $(\mathbf{NL}, \mathbf{NL})$

Suppose both luxury brands do not license (so that $D_f^a = D_f^b = 0$) and compete only in the snob market s. Then (1) reveals that a snob located at θ will obtain a net utility $U_s^A(\theta) = v_s - t_s\theta - p^A$ from purchasing A or $U_s^B(\theta) = v_s - t_s(1-\theta) - p^B$ from purchasing B. Hence, the marginal snob θ_s is indifferent between A and B, where $\theta_s = 1/2 + (p^B - p^A)/2t_s$. Because market s is fully covered, the proportion of snobs purchasing from brand A and B are $D_s^A = \theta_s$ and $D_s^B = 1 - \theta_s$, respectively. Substituting D_s^A, D_s^B, and $T^A = T^B = 0$ into (3), we obtain the profits of brand A and B. Then, by considering the first-order conditions, the optimal prices for the case when both brands do not license are equal to $p^A = p^B = c + t_s$. Consequently, brand A and B share market s equally so that $D_s^I = \frac{1}{2}$ for $I = A, B$. From (3), the brands' profits for the case when both do not license satisfy:

$$\Pi^I(NL, NL) = \frac{t_s}{2} \text{ for } I = A, B. \tag{9}$$

Throughout this paper, we use '(X, Y)' to denote the case when brand A chooses licensing strategy X and brand B chooses licensing strategy Y.

5.2. Both brands license via fixed-fee contracts $(\mathbf{F}, \mathbf{F})$

Consider the case when brands A and B license their brand names to licensees a and b by charging fixed fees $T^A = k^A$ and $T^B = k^B$, respectively. To begin, we examine consumers' purchasing decisions. First, a snob located at θ will obtain a net utility U_s^I from purchasing brand I's product, where:

$$U_s^A(\theta) = v_s - t_s\theta - \lambda_s\beta D_f^{a(e)} - p^A \text{ and } U_s^B(\theta) = v_s - t_s(1-\theta) - \lambda_s\beta D_f^{b(e)} - p^B.$$

[2] Through an extensive numerical study, we analyzed the case where markets are not fully covered and some consumers may prefer not to buy. We find that our main insights continue to hold.

Because market f is fully covered ($D_f^{b(e)} = 1 - D_f^{a(e)}$), the marginal snob θ_s who is indifferent between purchasing from brand A and B (i.e., $U_s^A(\theta_s) = U_s^B(\theta_s)$) is $\theta_s = \frac{1}{2} + \frac{p^B - p^A + \beta\lambda_s - 2\beta\lambda_s D_f^{a(e)}}{2t_s}$. Similarly, because market s is fully covered ($D_s^{B(e)} = 1 - D_s^{A(e)}$), the marginal follower θ_f who is indifferent between purchasing licensee a and b is given by: $\theta_f = \frac{1}{2} + \frac{p^b - p^a - \lambda_f + 2\lambda_f D_s^{A(e)}}{2t_f}$. By rational expectations, $D_s^{A(e)} = D_s^A = \theta_s$, and $D_f^{a(e)} = D_f^a = \theta_f$. This observation enables us to solve for θ_s and θ_f simultaneously, getting:

$$\theta_s = \frac{1}{2} + \frac{t_f\,(p^B - p^A) - \beta\lambda_s\,(p^b - p^a)}{2t_f t_s + 2\beta\lambda_f\lambda_s}, \qquad (10)$$

$$\theta_f = \frac{1}{2} + \frac{\lambda_f\,(p^B - p^A) + t_s\,(p^b - p^a)}{2t_f t_s + 2\beta\lambda_f\lambda_s}. \qquad (11)$$

Using (10) and (11), $T^I = k^I$ for $I = A, B$, $D_s^A = 1 - D_s^B$, and $D_f^a = 1 - D_f^b$ along with (3)-(4), we obtain profits of brands A and B and licensees a and b. By considering the first-order conditions simultaneously, we characterize the optimal prices for brand I ($I = A, B$) and its licensee i ($i = a, b$) as follows:

$$p^I = c + t_s + \frac{\beta\lambda_f\lambda_s}{t_f},\; p^i = t_f + \frac{\beta\lambda_f\lambda_s}{t_s}. \qquad (12)$$

Observe from (12) that the equilibrium prices are the same for both brands and for both licensees, and all prices are independent from the fixed-fee. Hence, for any fixed-fee k^I between brand I and its license i, the market shares of brand I and its licensee i satisfy: $D_s^I = 1/2$ for $I = A, B$ and $D_f^i = 1/2$ for $i = a, b$.

Next, relative to the no licensing case (NL, NL) presented in §5.1, (12) reveals that the equilibrium prices of both brands are 'higher' with licensing because $p^I > c + t_s$ for $I = A, B$. Furthermore, while the equilibrium price associated with the monopoly case (F) as stated in (7) is decreasing in λ_s, the brands' prices in the duopoly case are now increasing in the snobs' sensitivity to brand popularity λ_s. This key difference is due to the fact that, as both brands share market s equally in the duopoly case, the 'negative popularity effect' of each brand cancels each other out so that the net effect is absent.

Notice that the term $\beta\lambda_f\lambda_s/t_f$ in p^I as stated in (12) captures an 'indirect effect' of licensing that can soften competition in markets s and f [see, Cabral and Villas-Boas, 2005]. To elaborate, suppose brand A increases its price by one unit. Then brand B's market share in market s will increase, and this increase in popularity of brand B will make licensee b's product (that carries brand B's name) becomes more attractive to followers in market f (due to the followers' 'positive' popularity effect). Consequently, licensee a's sales will decrease, but it will increase the snobs' valuation of brand A in market s (due to the snobs' 'negative' popularity effect), which affords brand A to increase its price a little bit without affecting its demand. As competition between brand A and B in market s softens, both brands can afford to charge higher prices with licensing (than the case when no brand licenses). Furthermore, as followers' desire to adopt the brand or snobs' sensitivity to the brand popularity in market f increases, a unit increase in brand A's price has more impact on licensee a's market share or snobs' valuations. Consequently, the competition between brands softens more, and brands' prices increase in λ_s and λ_f. By using the same argument, the same indirect effect softens the competition between licensees so that both licensees charge higher prices as λ_s or λ_f increases (see equilibrium price p^i ($i = a, b$) in (12)).

5.2.1. Free licensing (FL). To isolate the effect of licensing without the influence of the license fee, let us study the case when both brands license their names for free so that $T^I = k^I = 0$ for $I = A, B$. First, using equilibrium prices given in (12), $D_s^I = 1/2$ and $T^I = 0$ for $I = A, B$ along with (3), each brand I's profit for the case when both brands license via fixed-fee contract is given by:

$$\Pi^I(FL, FL) = \frac{t_s}{2} + \frac{\beta\lambda_f\lambda_s}{2t_f} > \frac{t_s}{2} = \Pi^I(NL, NL) \text{ for } I = A, B, \tag{13}$$

where the last inequality follows from (9). This observation can be stated formally as follows:

LEMMA 1. *Suppose markets s and f are fully covered. Then, relative to the no licensing case, each brand earns more from licensing their brand names even for free, i.e., $\Pi^I(FL, FL) > \Pi^I(NL, NL)$ for $I = A, B$.*

Lemma 1 implies that, even with zero licensing fee, both brands earn more by licensing their brands to their respective licensees. This result is in contrast to the monopoly case (F) and appears to be counter-intuitive because licensing has a 'negative popularity effect' on snobs' valuations. However, as discussed above, in duopoly, the negative popularity effect of licensing is absent and the competition between brands is softened due to the 'indirect effect' across markets s and f. Consequently, both brands can afford to charge higher prices and obtain more profits in market s even when they license their brand names for free.

5.2.2. Optimal fixed fee $\mathbf{k^I}$. We now determine the optimal fixed licensing fees. By using the fact that licensees share market f equally so that $D_f^i = 1/2$ for $i = a, b$, the profit of licensee i ($i = a, b$), as stated in (4), can be simplified as $\Pi^i = p^i\beta/2 - k^I$ for any given fixed fee k^I. By using the fact that $p^i = t_f + \beta\lambda_f\lambda_s/t_s$ for $i = a, b$ as stated in (12), it is optimal for brand I to set the fixed fee equal to $k^I = p^i\beta/2$ to extract the entire surplus from its licensee. This observation implies that the optimal fixed fee satisfies:

$$k^I = \left(t_f + \frac{\beta\lambda_f\lambda_s}{t_s}\right)\frac{\beta}{2}.$$

Hence, brand I's profit ($I = A, B$) is equal to $\Pi^I(F, F) = \Pi^I(FL, FL) + k^I$ and it is given by:

$$\Pi^I(F, F) = \frac{t_s}{2} + \frac{\beta\lambda_f\lambda_s}{2t_f} + \left(t_f + \frac{\beta\lambda_f\lambda_s}{t_s}\right)\frac{\beta}{2} > \frac{t_s}{2} = \Pi^I(NL, NL), \tag{14}$$

where the last inequality follows from (9). Now, recall from Proposition 1 that, when a brand operates as a monopoly, the licensing decision depends on the snobs' sensitivity towards brand's popularity λ_s. However, in a duopoly setting, it is always beneficial for luxury brands to license their names because licensing can soften competition in the snob market so that brands can charge more and earn more.

5.3. Only one brand licenses by using a fixed-fee contract $(\mathbf{F}, \mathbf{NL})$

It remains to consider the case when exactly one brand licenses by using a fixed-fee contract. Because both brands and both licensees are symmetric, it suffices to consider the case (F, NL) in which brand A licenses its name to licensee a via a fixed-fee contract $T^A = k^A$ so that licensee a operates as a monopoly in market f. To begin, let us consider the snobs in market s. Because licensee a operates as a monopoly that covers the entire market f (i.e., $D_f^a = 1$), a snob located at θ can obtain net utilities $U_s^A(\theta) = v_s - t_s\theta - \lambda_s\beta - p^A$ and $U_s^B(\theta) = v_s - t_s(1-\theta) - p^B$ from purchasing brand A and B, respectively. Then, the marginal snob θ_s who is indifferent between purchasing A versus B is given by:

$$\theta_s = \frac{1}{2} + \frac{p^B - p^A - \lambda_s\beta}{2t_s}. \tag{15}$$

By anticipating that the demands for A and B are θ_s and $(1-\theta_s)$, respectively, the net utility to be obtained by a follower located at θ who purchases licensee a's product is equal to $U_f^a(\theta) = v_f - t_f\theta + \lambda_f\theta_s - p^a$. To ensure that the entire market f is covered by licensee a's product, it is optimal for licensee a to set its price $p^a = v_f - t_f + \lambda_f\theta_s$ so that the follower located at $\theta = 1$ will purchase its licensed product.

Using (15), the fact that $p^a = v_f - t_f + \lambda_f\theta_s$, $D_s^A = 1 - D_s^B = \theta_s$, $D_f^a = 1 - D_f^b = 1$, $T^A = k^A$ and $T^B = 0$ (as brand B does not license), we can use (3) and (4) to express the profit functions for brands A and B and the only licensee a as functions of p^A and p^B. Also, by considering the first-order conditions associated with the profit functions of brands A and B simultaneously (due to the underlying price competition between both brands in market s) and by considering the bounds associated with θ_s as stated in (15), we can determine the equilibrium price of both brands and the licensee a as follows.

Case 1: When $\lambda_s < 3t_s/\beta$. In this case, the optimal prices are given by:

$$p^A = c + t_s - \frac{\beta\lambda_s}{3},\ p^B = c + t_s + \frac{\beta\lambda_s}{3},\ p^a = v_f + \lambda_f\left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s}\right) - t_f. \tag{16}$$

Observe that $p^A < p^B$ when brand A licenses its brand name to licensee a. This result is due to the negative popularity effect. To stay competitive in market s, brand A has to charge a lower price than B.

Case 2: When $\lambda_s \geq 3t_s/\beta$. By considering (16), (15) reveals that $\theta_s = 1/2 - \beta\lambda_s/6t_s$. Hence, as the negative popularity effect is high (i.e., $\lambda_s \geq 3t_s/\beta$), $\theta_s = 0$ so that no snob will purchase brand A once it licenses its brand name to licensee a. Consequently, brand B operates as a monopoly in market s and licensee a operates as a monopoly in market f. While brand A's price p^A is irrelevant, brand B and licensee a will set their prices to ensure their respective markets are fully covered so that:

$$p^B = v_s - t_s,\ \ p^a = v_f - t_f.$$

By considering the selling price p^a in both cases 1 and 2, it is easy to check that licensee a's profit is equal to $p^a\beta - k^A$. Using the same argument as before, we can conclude that it is optimal for brand A to set its fixed fee k^A to extract all of licensee a's profit so that:

$$k^A = \left(v_f + \lambda_f\left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s}\right)^+ - t_f\right)\beta,$$

where $(x)^+ = \max(x, 0)$. Hence, by accounting for k^A, we can substitute the optimal prices p^A and p^B into (3) in both cases to retrieve the optimal profit for brand A that licenses and for brand B that does not license. Due to the fact that both brands are symmetric, we can determine the profit of each brand for the case when only brand B licenses as follows: $\Pi^B(NL, F) = \Pi^A(F, NL)$ and $\Pi^A(NL, F) = \Pi^B(F, NL)$. Thus, profits of brands when only one brand uses fixed-fee contract are given by:

$$\Pi^A(F, NL) = \Pi^B(NL, F) = 2t_s\left(\left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s}\right)^+\right)^2 + \left(v_f + \lambda_f\left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s}\right)^+ - t_f\right)\beta, \quad (17)$$

$$\Pi^B(F, NL) = \Pi^A(NL, F) = \begin{cases} 2t_s\left(\frac{1}{2} + \frac{\beta\lambda_s}{6t_s}\right)^2, & \text{if } \lambda_s < 3t_s/\beta, \\ v_s - t_s - c, & \text{if } \lambda_s \geq 3t_s/\beta. \end{cases} \quad (18)$$

We now compare brands' profits above against (9) (for the case (NL, NL)) and (14) (for the case (F, F)). Lemma 2 specifies the conditions under which it is beneficial for only one brand to license. In preparation, we define the threholds $\lambda_{sk}^{(1)}$ and $\lambda_{sk}^{(2)}$ as follows:

$$\lambda_{sk}^{(1)} = \begin{cases} \frac{2t_s + \beta\lambda_f - \sqrt{\beta^2\lambda_f^2 + 4t_s\left(t_s - 2\left(v_f - t_f\right)\beta\right)}}{2\beta/3}, & \text{if } v_f \leq t_f + t_s/2\beta, \\ \infty, & \text{if } v_f > t_f + t_s/2\beta. \end{cases} \quad (19)$$

$$\lambda_{sk}^{(2)} = \frac{3\lambda_f t_s - 2t_f t_s + 3\beta\lambda_f t_f + \sqrt{\left(3\lambda_f t_s - 2t_f t_s + 3\beta\lambda_f t_f\right)^2 + 4\beta t_f^3 t_s}}{2\beta t_f/3}, \quad (20)$$

and we let $\lambda_{sk}^{(3)} = \min(3t_s/\beta, \lambda_{sk}^{(2)})$.

LEMMA 2. *Suppose markets s and f are fully covered. Then:*

(i) If one brand does not license, it is always beneficial for the other brand to license via the fixed-fee contract (i.e., $\Pi^A(F, NL) > \Pi^A(NL, NL)$, or $\Pi^B(NL, F) > \Pi^B(NL, NL)$) if, and only if, $\lambda_s < \lambda_{sk}^{(1)}$.

(ii) If one brand licenses via a fixed-fee contract, it is always beneficial for the other brand to license via the fixed-fee contract (i.e., $\Pi^A(F, F) > \Pi^A(NL, F)$, or $\Pi^B(F, F) > \Pi^B(F, NL)$) if, and only if, $\lambda_s < \lambda_{sk}^{(3)}$.

Lemma 2(i) implies that, when followers in market f have high valuation v_f for the licensed product (i.e., when $v_f > t_f + \frac{t_s}{2\beta}$ so that $\lambda_{sk}^{(1)} = \infty > \lambda_s$ as observed from (19)), the licensed product will sell well in market f and the brand that licenses can command a high fixed fee from its licensee so that this licensee fee always outweighs the loss of profit from market s due to the negative popularity effect. Hence, relative to the case that no brand licenses, one of the brands will always deviate and license. However, as is also implied by Lemma 2(i), when followers' valuation of the licensed product is low (i.e., $v_f \leq t_f + \frac{t_s}{2\beta}$) and the snobs' negative popularity effect is high (i.e., $\lambda_s > \lambda_{sk}^{(1)}$), licensing is no longer beneficial. Lemma 2(ii) shows that, when the snobs' negative popularity effect λ_s is sufficiently low (i.e., $\lambda_s < \lambda_{sk}^{(3)}$) and when only brand A licenses, the brand that does not license (brand B) should also license to increase its profit. In this case, if brand B licenses in addition to brand A, then the competition is softened in market s due to the indirect effect as discussed earlier. Moreover, this indirect effect will also soften the competition in market f. By licensing, brand B loses some of its share in market s, but brand B can gain from a higher selling price (due to softened competition in market s) and gain from the license fee to be collected from licensee b. As it turns out, this gain outweighs the loss: brand B should also license when $\lambda_s < \lambda_{sk}^{(3)}$. In summary, Lemma 2 indicates that brands' equilibrium licensing strategies under fixed-fee contracts depend critically on snobs' and followers' sensitivity to brand popularity (λ_s and λ_f). We will examine this issue next.

5.4. Equilibrium licensing strategy under fixed-fee contracts

By comparing brands' profit functions (presented in the last section) under different licensing strategies as in a two-player simultaneous-move game, Proposition 2 characterizes the licensing strategy that each brand will adopt in equilibrium under fixed-fee contracts. In preparation, we define two additional thresholds $\lambda_{sk}^L = \min(\lambda_{sk}^{(1)}, \lambda_{sk}^{(3)})$ and $\lambda_{sk}^H = \max(\lambda_{sk}^{(1)}, \lambda_{sk}^{(3)})$, where $\lambda_{sk}^{(3)} = \min(3t_s/\beta, \lambda_{sk}^{(2)})$, and $\lambda_{sk}^{(1)}$ and $\lambda_{sk}^{(2)}$ are given, respectively, by (19) and (20).

PROPOSITION 2. *Suppose both markets s and f are fully covered. Then, under fixed-fee licensing contracts, the equilibrium licensing strategy to be adopted by both brands can be characterized as follows:*

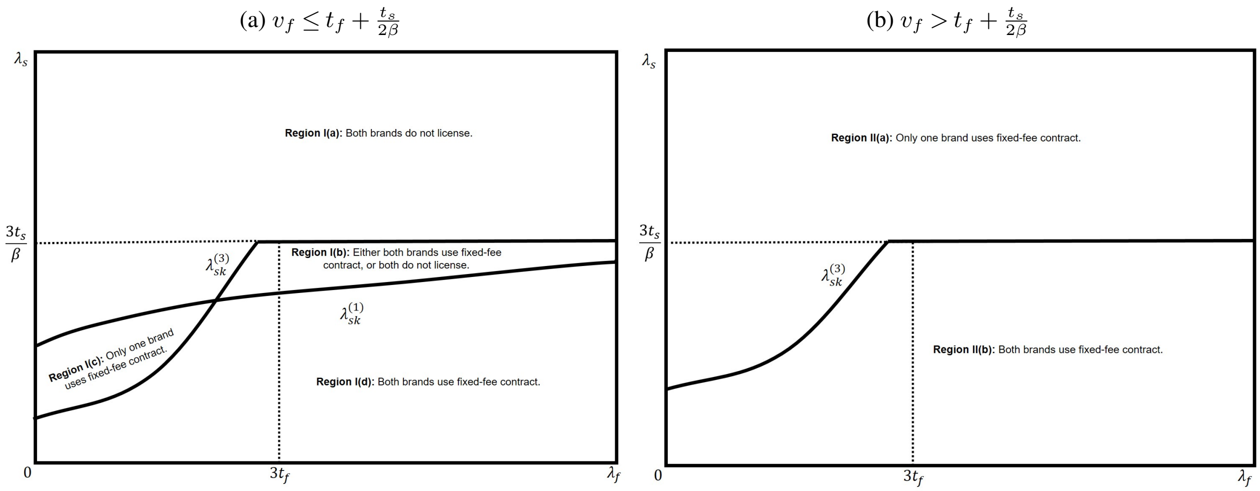
I. When the base valuation of the licensed product is low so that $v_f \leq t_f + \frac{t_s}{2\beta}$,
- *(a) both brands do not license if $\lambda_s \geq \lambda_{sk}^H$;*
- *(b) both brands either license or do not license (i.e., two equilibria exist) if $\lambda_{sk}^{(1)} < \lambda_s < \lambda_{sk}^{(3)}$;*
- *(c) only one brand licenses if $\lambda_{sk}^{(3)} < \lambda_s < \lambda_{sk}^{(1)}$;*
- *(d) both brands license if $\lambda_s \leq \lambda_{sk}^L$.*

II. When the base valuation of the licensed product is high so that $v_f > t_f + \frac{t_s}{2\beta}$,
- *(a) only one brand licenses if $\lambda_s > \lambda_{sk}^{(3)}$;*
- *(b) both brands license if $\lambda_s \leq \lambda_{sk}^{(3)}$.*

Figure 1a illustrates Proposition 2(I) for the case when $v_f \leq t_f + \frac{t_s}{2\beta}$. Specifically, Proposition 2(Ia) shows that both brands do not license for sufficiently high λ_s values ($\lambda_s \geq \lambda_{sk}^H$, i.e., region I(a)). This result is in-congruent with Proposition 1 for the monopoly model stating that the monopoly should not license when the negative popularity effect for snobs (λ_s) is too high. However, recall from Lemma 1 that, due to the 'indirect effect' of licensing that can soften competition, licensing is more profitable for the brands as the snobs' negative popularity effect λ_s or the followers' positive popularity effect λ_f increases. Even so, it is interesting to observe that no brand should license when λ_s lies within region I(a). To understand why, recall from Lemma 1 that each brand would be better off if both brands could 'commit' to licensing via fixed-fee contracts. However, in the absence of such a commitment, brands face a prisoner's dilemma and do not license. We formalize this argument in Corollary 1.[3] To better understand the intuition behind this

[3] The proof of Corollary 1 follows from Proposition 2(Ia) by comparing different payoffs using (13). We omit the details for brevity.

Figure 1 Brands' Equilibrium Licensing Strategies under Fixed-Fee Contracts



corollary, consider a scenario where both brands license. In region I(a), the negative popularity effect λ_s is significantly high so that a brand is better off by making its brand more exclusive to please the snobs. In this case, at least one brand will want to deviate and not to license (e.g., $\Pi^B(F, NL) > \Pi^B(F, F)$ by Lemma 2(ii)). Moreover, if one of the brands deviates and does not license, the profit of the other brand significantly decreases due to the negative popularity effect (e.g., $\Pi^A(F, NL) < \Pi^A(NL, NL)$ by Lemma 2(i)). Hence, both brands end up not licensing in equilibrium in region I(a).

COROLLARY 1. *Suppose markets s and f are fully covered. In equilibrium under fixed-fee contracts, when $v_f \leq t_f + \frac{t_s}{2\beta}$, if $\lambda_s \geq \lambda_{sk}^H$, brands face prisoner's dilemma and both do not license even though each brand would be better off if both brands would license via fixed-fee contracts.*

Figure 1a also shows that, given any $\lambda_s < 3t_s/\beta$, when followers' desire to adopt λ_f is significantly high (region I(b)), cases where both brands use fixed-fee contracts also become an equilibrium. In region I(b), licensing revenues are significant (due to high λ_f), but the direct negative impact is also significant (due to high λ_s). The latter dominates the former and licensing decreases a brand's profits when the other brand does not license (i.e., $\Pi^A(NL, NL) > \Pi^A(F, NL)$ by Lemma 2(i)). On the other hand, the former dominates the latter and a brand benefits from licensing when the other brand also licenses (i.e., $\Pi^B(F, F) > \Pi^B(F, NL)$ by Lemma 2(ii)) since the negative popularity effect is zero under competition, and the indirect effect softens competition in both markets and improves profits when both brands license. Thus, there are two equilibria in region I(b), i.e., no brand licenses or both brands license.

Next, Proposition 2(Id) shows that both brands would license for sufficiently low λ_s values ($\lambda_s \leq \lambda_{sk}^L$, i.e., region I(d) in Figure 1a). This is because, in region I(d), licensing always benefits a brand independent from whether the other brand licenses or not (i.e., $\Pi^A(F, F) > \Pi^A(NL, F)$ and $\Pi^A(F, NL) > \Pi^A(NL, NL)$

by Lemma 2). This is because licensing softens competition in both markets (due to indirect effect) and/or provides significant additional revenues.

Figure 1a also shows that, for a given λ_s in region I(c), only one brand licenses if followers' sensitivity to brand popularity is sufficiently low (region I(c)). A brand benefits from licensing in region I(c) only when the other brand does not license (i.e., $\Pi^A(F,F) < \Pi^A(F,NL)$ and $\Pi^A(F,NL) > \Pi^A(NL,NL)$ by Lemma 2). In such cases, licensing revenues are not high enough (due to low λ_f) for both brands to license and licensing decreases their exclusivity. Instead, only one brand licenses and obtains lower profits from market s, yet its total profits increase as it receives all licensing revenues. However, when licensing revenues are sufficiently high (e.g., high β so that the size of market f is sufficiently large), it is never the case that only one brand licenses in equilibrium (i.e., $\lambda_{sk}^{(3)} = 3t_s/\beta$) and region I(c) in Figure 1a disappears. This is formalized in the following corollary.

COROLLARY 2. *Suppose markets s and f are fully covered. In equilibrium under fixed-fee contracts, when followers' base valuation and the size of market f are low (i.e., $v_f \le t_f + \frac{t_s}{2\beta}$ and $\beta \ge 3t_s/t_f$), there is no equilibrium in which only one brand licenses.*

Lastly, Figure 1b illustrates Proposition 2 when followers' valuation for the licensed product is sufficiently high ($v_f > t_f + \frac{t_s}{2\beta}$). When $v_f > t_f + \frac{t_s}{2\beta}$, (19) states that $\lambda_{sk}^{(1)} = \infty$. Hence, by Lemma 2 asserts that it is always beneficial for a brand to use a fixed-fee contract when the other brand does not license, i.e., $\Pi^A(F,NL) > \Pi^A(NL,NL)$. Therefore, as is shown in Figure 1b, for $v_f > t_f + \frac{t_s}{2\beta}$, unlike in the monopoly model, licensing is always optimal and at least one brand uses a fixed-fee contract in equilibrium. Specifically, both brands license by using fixed-fee contracts when the snobs' negative popularity effect λ_s is low (i.e., $\lambda_s < \lambda_{sk}^{(3)}$), and only one brand licenses via a fixed-fee contract otherwise.

6. Royalty Contracts (R)

We now extend our analysis to the case when both brands consider licensing using royalty contracts under which each brand I ($I = A, B$) charges its respective licensee i ($i = a, b$) a license fee $T^I = r^I \beta D_f^i$ that depends on the demand in market f, where the 'royalty fee per unit sold' r^I is determined by brand I.[4] For brevity, we focus on the duopoly case, and the analysis of the monopoly case is deferred to Appendix C.[5] We use the same approach as presented in §5 to analyze the duopoly case under a royalty contract. Because

[4] In practice, the royalties are based on a percentage of the licensee's overall revenue, and this percentage $\alpha^I \le 1$ is specified by brand I ($I = A, B$). In this case, the transfer payment between brand I and its licensee i ($i = a, b$) is equal to $T^I = \alpha p^i \beta D_f^i$. In line with the literature [e.g., Kamien and Tauman, 1986; Wang, 1998; Poddar and Sinha, 2002], we assume that royalties are collected for each unit sold. However, by letting $\alpha^I = r^I/p^i$ for $I = A, B$ and $i = a, b$, it is easy to check that both models are equivalent in our setting.

[5] Relative to a fixed-fee contract, we find that the monopolist brand licenses in a wider range of the negative popularity effect λ_s under a royalty contract because it enables the brand to 'indirectly control' its licensee's price (and its demand). Consequently, armed with this indirect control, the royalty contract dominates the fixed-fee contract when λ_s is sufficiently high. See Appendix C for further details.

analysis of the case when both brands do not license (NL, NL) is the same as presented in Section 5.1, it suffices to consider only two cases: (a) both brands use royalty contracts to license; and (b) one brand (brand A) uses a royalty contract to license and the other brand does not license. While we use the same approach to analyze different settings as before, we omit some details to avoid repetition.

6.1. Both brands license by using royalty contracts $(\mathbf{R}, \mathbf{R})$

Consider the case where brands A and B license their brand names to licensees a and b by charging (per-unit) royalty fees r^A and r^B, respectively. To ensure that equilibrium royalty fees and prices exist,[6] we assume that the followers' positive popularity effect λ_f is sufficiently high so that:

$$\lambda_f \geq \sqrt{\frac{t_f t_s}{2\beta}}. \tag{21}$$

By using the same approach as presented in §5.2, we can characterize the equilibrium royalty fee r^I and prices in Lemma 3.

LEMMA 3. *Suppose that markets s and f are fully covered and condition* (21) *holds. Then, when both brands license by using royalty contracts, the royalty fee r^I satisfies:*

$$r^I = \begin{cases} 0, & \text{if } \lambda_f \geq 3t_f \text{ and } \lambda_s < \lambda_{sr}^{(1)}, \\ t_f \frac{2\lambda_s\left(2\beta\lambda_f + t_s\right) + 3t_s\left(3t_f - \lambda_f\right)}{3t_f t_s + 2\beta\lambda_f^2}, & \text{if otherwise,} \end{cases}$$

where $\lambda_{sr}^{(1)}$ is given by (41) in Appendix B.1. Also, the optimal prices of brand I ($I = A, B$) and its licensee i ($i = a, b$) satisfy:

$$p^I = c + t_s + \frac{\beta\lambda_f\lambda_s}{t_f} - \frac{\beta\lambda_f r^I}{t_f},\ p^i = r^I + t_f + \frac{\beta\lambda_f\lambda_s}{t_s}.$$

Relative to brand I's equilibrium prices under the fixed-fee contract as stated in (12), Lemma 3 reveals that, under royalty contracts, brands will charge lower prices in market s, while their licensees charge higher prices in market f. To understand why, observe first that the 'indirect effect' (i.e., $\beta\lambda_f\lambda_s/t_f$) that softens the competition in market s continues to persist under royalty contracts. However, under the royalty contracts, the royalties to be collected by each brand depend on the demand of the respective licensed product in market f. At the same time, due to the the followers' positive popularity effect λ_f, one can boost the demand of the licensed product in market f by increasing the demand of the brand in market s. For these reasons, each brand has an incentive to lower its price p^I to increase its demand in market s (which causes the licensed product's demand in market f to increase). This price-lowering strategy is caused by the royalties (that depend on the sales of the licensed product in market f), and we shall refer to this effect

[6] We make this assumption for ease of exposition, and the condition in (21) is sufficient but not necessary and there might cases where that condition is violated but the equilibrium still exists. The condition in (21) is equivalent to assuming the size of market f being sufficiently big (high β). Therefore, our assumption is in line with practice as the size of licensing market (market f) is much larger compared to the size of the market for a brand's own luxury goods (market s).

(i.e., $r^I\beta\lambda_f/t_f$) as the 'royalty effect' that intensifies price competition between brands in market s so that each brand charges a lower price under royalty contracts. To make up for the lower profit margin according to the term $\beta\lambda_f r^I/t_f$ in market s, each brand can leverage its direct control to push the licensee to increase its price according to an extra term r^I, which is determined by brand I.[7]

By using the royalty fee and the equilibrium prices presented in Lemma 3, $T^I = r^I\beta D_f^i$ and $D_s^I = 1/2$ for $I = A, B$, and $D_f^i = 1/2$ for $i = a, b$ into (3), brand I's profit satisfies:

$$\Pi^I(R,R) = \frac{t_s}{2} + \frac{\beta\lambda_f\lambda_s}{2t_f} + \frac{\beta}{2}r^I\left(1 - \frac{\lambda_f}{t_f}\right) \text{ for } I = A, B. \tag{22}$$

By comparing brands' profits as stated in (22) against (9) (as in the no licensing case (NL, NL)) and against (14) (as in the case (F, F) under fixed-fee contracts), we obtain Lemma 4 that involves different threshold values for λ_s (namely, $\lambda_{sr}^{(2)}$ and $\lambda_{sr}^{(3)}$ that are given, respectively, by (42) and (43) in Appendix B.1) and threshold values for λ_f (namely, $\lambda_{fr}^L = \min(\lambda_{fr}^{(1)}, \lambda_{fr}^{(2)})$ and $\lambda_{fr}^H = \max(\lambda_{fr}^{(1)}, \lambda_{fr}^{(2)})$, where $\lambda_{fr}^{(1)}, \lambda_{fr}^{(2)} < t_f$ are given, respectively, by (45) and (46) in Appendix B.1).

LEMMA 4. *Suppose markets s and f are fully covered. Then:*

(i) Relative to the case when both brands license via royalty contracts, as in case (R, R), each brand I ($I = A, B$) is better off when both brands do not license (i.e., $\Pi^I(NL, NL) > \Pi^I(R, R)$ for $I = A, B$) if, and only if, $\lambda_s < \lambda_{sr}^{(2)}$ and $\lambda_f \in (t_f, 3t_f)$.

(ii) Relative to the case that both brands license via royalty contracts, as in case (R, R), each brand I ($I = A, B$) earns more profits in the case that both brands license via fixed-fee contracts (i.e., $\Pi^I(F, F) > \Pi^I(R, R)$ for $I = A, B$) if, and only if: (1) $\lambda_s > \lambda_{sr}^{(3)}$ when $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$ and $\lambda_{fr}^{(1)} \le \lambda_{fr}^{(2)}$; or (2) $\lambda_s < \lambda_{sr}^{(3)}$ when $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$ and $\lambda_{fr}^{(1)} > \lambda_{fr}^{(2)}$; or (3) $\lambda_f \ge \lambda_{fr}^H$.

Lemma 4(i) asserts that, instead of licensing via royalty contracts, as in the case (R, R), both brands are better off from not licensing when $\lambda_f \in (t_f, 3t_f)$ and $\lambda_s < \lambda_{sr}^{(2)}$ so that the royalty effect ($r^I\beta\lambda_f/t_f$) dominates the indirect effect of licensing ($\beta\lambda_f\lambda_s/t_f$). Hence, licensing via royalty contracts is not beneficial. Clearly, both brands can eliminate the 'royalty effect' by licensing for free ($r^A = r^B = 0$) so that they can benefit from the 'indirect effect' of licensing. In fact, because $\lambda_f > t_f$, (22) reveals that each brand would be better off in equilibrium if both could 'commit' to license its name for free by setting $r^A = r^B = 0$. Because such a commitment is absent, brands face a prisoner's dilemma and both will charge a positive royalty fee (i.e., $r^A = r^B > 0$ by Lemma 3) and end up with significantly lower profits. Consequently, both brands are better off not licensing in equilibrium.

Lemma 4(ii) implies that the royalty contract performs better when followers' brand appreciation is sufficiently low (i.e., $\lambda_f \le \lambda_{fr}^L$). The underlying intuition can be described as follows. First, consider the

[7] In the event when λ_f is high while λ_s is low (i.e., $\lambda_f \ge 3t_f$ and $\lambda_s < \lambda_{sr}^{(1)}$), the royalty effect is much stronger than the indirect effect. Consequently, each brand sets $r^I = 0$ to eliminate the royalty effect and licenses for free.

extreme case where followers do not appreciate the brand, i.e., $\lambda_f = 0$. In this case, one can check from (12) and Lemma 3 that the brand's prices under fixed-fee and royalty contracts are equal to $p^I = c + t_s$ $(I = A, B)$. At the same time, one can also check from (14) and Lemma 3 that each brand collects licensing revenue $t_f\beta/2$ and $(r^I + t_f)\beta/2$ under the fixed-fee and royalty contracts. Hence, brands collect more licensing revenues from market f and they are better off under royalty contracts when λ_f is sufficiently small.

Next, consider the other extreme case where followers' brand appreciation is significant, i.e., $\lambda_f = \infty$. In this case, the royalty effect $r^I\beta\lambda_f/t_f$ is significant and brands license for free ($r^A = r^B = 0$ by Lemma 3) under royalty contracts to eliminate this royalty effect. However, under fixed-fee contracts, (14) reveals that the fixed fee k^I is increasing in λ_f. Hence, when λ_f is sufficiently high, both brands earn more under fixed-fee contracts than licensing for free under royalty contracts. This explains case 3 in Lemma 4(ii).

6.2. Only one brand licenses by using a royalty contract $(\mathbf{R}, \mathbf{NL})$

Consider the case where only one brand (brand A) licenses via a royalty contract, while the other brand (brand B) does not license. Hence, brand A charges the royalty fee r^A for each unit that licensee a sells (as a monopoly) in market f. Before we present our analysis, let us make two observations. First, because the market f is fully covered by licensee a's product, we can use the same argument presented in §5.3 to show that it is optimal for licensee a to set $p^a = v_f + \lambda_f\theta_s - t_f$. Second, because licensee a operates as a monopoly in market f that is fully covered, it is always optimal for brand A to set the royalty fee $r^A = p^a$ to extract licensee a's entire profit. By noting that these two observations are the same as presented in §5.3, we can conclude that, when only one brand licenses and the other does not, fixed-fee and royalty contracts are equivalent. Thus, the equilibrium prices when only one brand uses royalty contract are identical to those in §5.3, and the brands' profits when only one firm licenses via a royalty contract are given by:

$$\Pi^I(R, NL) = \Pi^I(F, NL), \tag{23}$$

$$\Pi^I(NL, R) = \Pi^I(NL, F) \tag{24}$$

for $I = A, B$, where $\Pi^I(F, NL)$ and $\Pi^I(NL, F)$ are given, respectively, by (17) and (18).

We now compare the profits when only one brand licenses by using a royalty contract, as in the case (R, NL) against the profits stated in (9) (as in the case of no licensing (NL, NL)) and the profits stated in (22) (as in the case when both brands license (R, R)), and present our result in Lemma 5,[8] where $\lambda_{sk}^{(1)}$ is given by (19), and we let $\lambda_{sr}^{(4)} = \min(3t_s/\beta, \lambda_{sr}^{(5)})$ with $\lambda_{sr}^{(5)}$ as given by (44) in Appendix B.1.

LEMMA 5. *Suppose markets s and f are fully covered.*

[8] We cannot show Lemma 5(ii) analytically when $t_f < \lambda_f < 3t_f$ and $\lambda_s < 3t_s/\beta$. In such cases, several numerical examples confirm that there are similar threshold λ_s values less than $3t_s/\beta$ that characterize cases where a brand is better off not licensing when the other brand uses a royalty contract. For brevity, we do not present these numerical examples in this paper.

(i) If one brand does not license, then it is always beneficial for the other brand to license via a royalty contract (i.e., $\Pi^A(R, NL) > \Pi^A(NL, NL)$ and $\Pi^B(NL, R) > \Pi^B(NL, NL)$) if, and only if, $\lambda_s < \lambda_{sk}^{(1)}$.

(ii) If one brand licenses via a royalty contract, then it is always beneficial for the other brand not to license (i.e., $\Pi^A(NL, R) > \Pi^A(R, R)$ and $\Pi^B(R, NL) > \Pi^B(R, R)$): (i) if, and only if, $\lambda_s \geq \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$; (ii) if $\lambda_s \geq 3t_s/\beta$ when $t_f < \lambda_f < 3t_f$; and (iii) if, only if, $\lambda_s \geq 3t_s/\beta$ when $\lambda_f \geq 3t_f$.

Because the profit function under a royalty contract are identical to that under a fixed-fee contract when only one brand licenses (as shown in (23) and (24)), Lemma 5(i) is akin to Lemma 2(i) so that the same intuition applies. Lemma 5(ii) shows that that in the case when both brands license via royalty contracts, one of the brands always wants to deviate and not to license if snobs are sufficiently sensitive to brand popularity in market f (e.g., $\lambda_s \geq 3t_s/\beta$). Hence, when the negative popularity effect λ_s is very high, a brand is better off not licensing (by making its brand more exclusive). This implies that, in equilibrium, under royalty contracts, at most one brand will license for sufficiently high λ_s values. We will characterize brands' equilibrium licensing strategies under royalty contracts next.

6.3. Equilibrium licensing strategy under royalty contracts

By using the same approach as presented in §5.4, we characterize equilibrium licensing strategies under the royalty contract by comparing each brand's payoffs associated with different cases presented in this section. We summarize our results in Proposition 3. In preparation, we define $\lambda_{sr}^L = \min\left(\lambda_{sk}^{(1)}, \lambda_{sr}^{(4)}\right)$ and $\lambda_{sr}^H = \max\left(\lambda_{sk}^{(1)}, \lambda_{sr}^{(4)}\right)$, where $\lambda_{sr}^{(4)} = \min\left(3t_s/\beta, \lambda_{sr}^{(5)}\right)$, and $\lambda_{sk}^{(1)}$ and $\lambda_{sr}^{(5)}$ are given, respectively, by (19), and (44) in Appendix B.1.

PROPOSITION 3. *Suppose markets s and f are fully covered. Then, under royalty contracts, the equilibrium licensing strategy to be adopted by the brands can be characterized as follows:*

I. When the base valuation of the licensed product is low so that $v_f \leq t_f + \frac{t_s}{2\beta}$,

(a) both brands do not license if $\lambda_s \geq \lambda_{sr}^H$ when $\lambda_f \leq t_f$, or $\lambda_s \geq 3t_s/\beta$ when $\lambda_f > t_f$;

(b) both brands either license or do not license (i.e., two equilibria exist) if $\lambda_{sk}^{(1)} < \lambda_s < \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_{sk}^{(1)} < \lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$;

(c) only one brand licenses if $\lambda_{sr}^{(4)} < \lambda_s < \lambda_{sk}^{(1)}$ when $\lambda_f \leq t_f$;

(d) both brands license if $\lambda_s < \lambda_{sr}^L$ when $\lambda_f \leq t_f$, or $\lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$.

II. When the base valuation of the licensed product is high so that $v_f > t_f + \frac{t_s}{2\beta}$,

(a) only one brand licenses if $\lambda_s > \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_s > 3t_s/\beta$ when $\lambda_f \geq t_f$;

(b) both brands license if $\lambda_s < \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$.

In the same spirit as Proposition 5 associated with the monopoly case, Proposition 3(Ia) implies that both brands should not license under royalty contracts when the snobs' negative popularity effect is very strong (i.e., $\lambda_s \geq 3t_s/\beta$) because both brands cannot afford to dilute their brands via licensing. Also, coupled with

Lemma 4, Proposition 3(Ia) implies that, in some cases, for sufficiently high λ_s values (e.g., $\lambda_s \geq 3t_s/\beta$ when $\lambda_f \leq t_f$ or $\lambda_f \geq 3t_f$), both brands would actually be better off if they were able to commit to licensing via royalty contracts. However, without such a commitment, both brands face prisoner's dilemma and both end up not licensing. Akin to Corollary 1 associated with fixed-fee contracts, we formalize this result in Corollary 3 below.

COROLLARY 3. *Suppose markets s and f are fully covered. In equilibrium under royalty contracts, for $v_f \leq t_f + \frac{t_s}{2\beta}$, if $\lambda_s \geq 3t_s/\beta$ when $\lambda_f \leq t_f$ or $\lambda_f \geq 3t_f$, brands face prisoner's dilemma and do not license, even though each would be better off if both brands licensed via royalty contracts.*

Essentially, Proposition 3 possesses the same structure as presented in Proposition 2. Hence, the equilibrium licensing strategies under royalty contracts have a similar characteristics as under fixed-fee contracts (i.e., as illustrated in Figure 1). Hence, Proposition 3 can be interpreted in same manner as Proposition 2. We omit the details to avoid repetition.

7. Umbrella Branding (U)

We now extend our analysis to examine why some luxury brands extend in new product categories through umbrella branding while others prefer licensing, and to explore how competition and popularity effects can influence these extension strategies in equilibrium. To do so, we consider a 'centralized setting' in which brand I ($I = A, B$) 'owns its licensee' i ($i = a, b$) and decides on the prices in both markets s and f so as to maximize the total profit to be obtained from both markets. In line with Amaldoss and Jain [2015], umbrella branding involves extensions in new product categories, we continue to assume that market s and f are 'separate' in the sense that snobs in market s will not buy the extended products in market f, and followers in market f will not buy the luxury product in market s. We also continue to normalize the unit cost of each brand's low-quality product in market f to zero.[9] With a slight abuse of notation, we will use p^A to denote the price of the high-quality product of brand A in market s and p^a to denote the price of the low-quality product of brand A in market f. Similarly, we define p^B and p^b as prices of brand B in markets s and f, respectively. By noting that brand I sets both p^I and p^i in both markets under umbrella branding, the profit of brand $I = A, B$ can be written as:

$$\Pi^A\left(p^A, p^a\right) = (p^A - c)D_s^A + p^a\beta D_f^a, \tag{25}$$

$$\Pi^B\left(p^B, p^b\right) = (p^B - c)D_s^B + p^b\beta D_f^b. \tag{26}$$

[9] Here, we implicitly assume that both brands and licensees have the same cost structure for producing the low-quality product for market f. Also, while we ignore licensing firms' expertise in designing and manufacturing these products, we can incorporate these cost factors into our model of umbrella branding without changing our analysis significantly. If we were to incorporate these cost factors, the implication is that brands will charge higher prices in market f (i.e., higher p^a and p^b) and hence their profits will be lower. Consequently, licensing will dominate umbrella branding in a wider range of λ_s and λ_f values.

Again, we focus on the duopoly case by considering a similar sequence of events and by using the same approach as presented in §5. (The analysis of the monopoly case is presented in Appendix D).[10] We now analyze the umbrella branding strategy. Because the case when both brands do not use umbrella branding (i.e., do not extend in market f) is identical to the case when both brands do not license (NL, NL) as presented in §5.1, it suffices to study only two cases: (a) both brands use umbrella branding strategy to extend in market f; and (b) only one brand (brand A) uses umbrella branding. Also, when we compare licensing and umbrella branding in the duopoly setting, we will only consider licensing via fixed-fee contracts for brevity. However, when all other forms of licensing contracts (e.g., royalty contract) are taken into account, licensing will dominate umbrella branding in a wider range of λ_s and λ_f.

7.1. Both brands use umbrella branding $(\mathbf{U}, \mathbf{U})$

When both brands use umbrella branding and extend in market f (as examined in Amaldoss and Jain [2015, Section 5.2]), we can use a similar approach as presented in §5.2 to characterize prices of each brand in market s and f (p^I $(I = A, B)$ and p^i $(i = a, b)$) as follows:

$$p^I = c + t_s - \beta\lambda_f,\; p^i = t_f + \lambda_s. \tag{27}$$

First, observe from (27) that, under the umbrella branding strategy, brands' prices p^I in market s are decreasing in followers' sensitivity to brand popularity (λ_f). Hence, relative to the case when both brands do not extend, as in the case (NL, NL), there is an 'indirect effect' of umbrella branding that intensifies competition between brands so that $p^I < c + t_s$. To elaborate, each brand aims to maximize its total profit generated from both markets s and f under umbrella branding. Hence, as followers' positive popularity effect λ_f increases, each brand has an incentive to increase its demand in market s in order to sell more in market f (see Amaldoss and Jain [2015] for further details). Consequently, umbrella branding intensifies competition in market s, which triggers both brands to lower their prices in market s. This effect is in direct contrast to the 'indirect effect' of fixed-fee contracts that softens competition as discussed in §5.2.

Second, observe from (27) that prices of both brands are equal. Hence, each brand has equal market share in markets s and f so that $D_s^I = 1/2$ for $I = A, B$ and $D_f^i = 1/2$ for $i = a, b$. Applying this observation and (27), the profit of each brand under umbrella branding as stated in (25) and (26) can be expressed as:

$$\Pi^I(U,U) = \frac{t_s}{2} + \beta\frac{\lambda_s - \lambda_f + t_f}{2} \;\text{ for }\; I = A, B. \tag{28}$$

By comparing the profit $\Pi^I(U,U)$ given in (28) against those in cases (NL, NL) and (F, F) as presented in §5.1 and §5.2, we get Lemma 6.

[10] In the monopoly setting, umbrella branding is actually the first best as the brand's objective is to maximize total profits from both markets s and f. As the monopolist brand strikes the balance between the profits obtained from both markets, we show that umbrella branding yields a higher profit than licensing through a fixed-fee or royalty contract.

LEMMA 6. *Suppose markets s and f are fully covered. Then:*

(i) Relative to the case when no brand extends, as in case (NL,NL), each brand earns more when both brands extend via umbrella branding (i.e., $\Pi^I(U,U) > \Pi^I(NL,NL)$ for $I=A,B$) if, and only if, $\lambda_f < \lambda_s + t_f$.

(ii) Relative to the case when both brands license via fixed-fee contracts, as in case (F,F), each brand earns more when both brands extend via umbrella branding (i.e., $\Pi^I(U,U) > \Pi^I(F,F)$ for $I=A,B$) if, and only if, $\lambda_f < \lambda_{fc}^{(1)}$, where $\lambda_{fc}^{(1)}$ is given by (47) in Appendix B.2.

Lemma 6(i) implies that brands are better off not extending via umbrella branding when the followers' positive popularity effect is sufficiently high, i.e., $\lambda_f > \lambda_s + t_f$. This is because, when λ_f is high, the indirect effect of umbrella branding will intensify the competition between brands in market s even further so that each brand has to lower its price significantly as stated in (27). Consequently, both brands are better off not extending their brands in market f.

Interestingly, Lemma 6(ii) implies that brands are better off from licensing via fixed-fee contract instead of using umbrella branding to extend in market f when followers' desire to adopt the brand is sufficiently high (i.e., $\lambda_f > \lambda_{fc}^{(1)}$). This result is counter-intuitive and it is in direct contrast to that in the monopoly model, where umbrella branding is the first best as the monopolist brand aims to maximize total profits from both markets (see Appendix D). In the monopoly model, the brand prefers a more centralized control of the sales in market f due to the negative popularity effect. However, this effect cancels out in the duopoly model. Moreover, when the followers' positive popularity effect λ_f is sufficiently high, the competition between brands in market s is very intense due to the 'indirect effect' of umbrella branding, as explained above. However, under the fixed-fee contracts, competition is softened so that both brands can afford to charge higher prices and obtain higher profits. Therefore, a more centralized control of market f through umbrella branding is not always beneficial for brands under competition, and they are better off from a decentralized system where they use fixed-fee contracts to license, especially when λ_f is sufficiently high.

7.2. Only one brand uses the umbrella branding $(\mathbf{U},\mathbf{NL})$

Next, let us consider the case where only one brand (brand A) uses umbrella branding to extend in market f while the other brand (brand B) does not extend. Using the same approach as in the case (F,NL) as presented in §5.3, three cases emerge: (1) $\lambda_s \le (\lambda_f - 3t_s/\beta)^+$, (2) $(\lambda_f - 3t_s/\beta)^+ < \lambda_s < \lambda_f + 3t_s/\beta$, and (3) $\lambda_s \ge \lambda_f + 3t_s/\beta$.

Case 1: When $\boldsymbol{\lambda_s < (\lambda_f - 3t_s/\beta)^+}$. This case can possibly occur only when $\lambda_f > 3t_s/\beta$. When this case occurs, brand A is a monopoly in both markets, and brand B sells nothing and earns nothing. Prices and profits of brand A are the same as those in the monopoly case, as presented in Appendix D for fully-covered market$f$ (which occurs when v_f is sufficiently high).

Case 2: When $(\lambda_\mathbf{f} - \mathbf{3t_s}/\beta)^+ \le \lambda_\mathbf{s} < \lambda_\mathbf{f} + \mathbf{3t_s}/\beta$. When this case occurs, brand A sells in both markets and brand B competes with brand A only in market s. It can be shown that the optimal prices are given by:

$$p^A = c + t_s - \frac{\beta(2\lambda_f + \lambda_s)}{3},\ p^B = c + t_s + \frac{\beta(\lambda_s - \lambda_f)}{3},\ p^a = v_f + \lambda_f\left(\frac{1}{2} + \frac{\beta(\lambda_f - \lambda_s)}{6t_s}\right) - t_f.$$

It is easy to check from above that the brand that uses umbrella branding to extend (brand A) charges a lower price ($p^A < p^B$). While brand A suffers from a lower profit in market s, it gains an additional profit from market f due to its extension under umbrella branding.

Case 3: When $\lambda_\mathbf{s} \ge \lambda_\mathbf{f} + \mathbf{3t_s}/\beta$. When this case occurs, the snobs' negative popularity effect λ_s is very high. Therefore, the brand (brand A) that extends in market f will alienate the snobs and lose it entire market s to brand B completely (i.e., $\theta_s = 0$). Consequently, brand B is a monopoly in market s with price $p^B = v_s - t_s$, and brand A is a monopoly in market f with price $p^a = v_f - t_f$.

By using above prices and by using the fact that both brands are symmetric, we can use (25) and (26) to compute each brand's profit when only one of them uses the umbrella branding as follows:

$$\Pi^A(U,NL) = \Pi^B(NL,U) = \begin{cases} v_s - t_s - c + \beta\left(v_f + \lambda_f - \lambda_s - t_f\right), & \text{if } \lambda_s \le \lambda_f - \frac{3t_s}{\beta}, \\ \frac{t_s}{2}\left(1 + \frac{\beta(\lambda_f - \lambda_s)}{3t_s}\right)^2 + \beta\left(v_f - t_f\right), & \text{if } \lambda_f - \frac{3t_s}{\beta} < \lambda_s < \lambda_f + \frac{3t_s}{\beta}, \\ \beta\left(v_f - t_f\right), & \text{if } \lambda_s \ge \lambda_f + \frac{3t_s}{\beta}, \end{cases} \tag{29}$$

$$\Pi^B(U,NL) = \Pi^A(NL,U) = \begin{cases} 0, & \text{if } \lambda_s \le \lambda_f - \frac{3t_s}{\beta}, \\ \frac{t_s}{2}\left(1 - \frac{\beta(\lambda_f - \lambda_s)}{3t_s}\right)^2, & \text{if } \lambda_f - \frac{3t_s}{\beta} < \lambda_s < \lambda_f + \frac{3t_s}{\beta}, \\ v_s - t_s - c, & \text{if } \lambda_s \ge \lambda_f + \frac{3t_s}{\beta}. \end{cases} \tag{30}$$

Comparing brands' profits against those when no brand extends in market f, as in case (NL,NL), and when both brands extend by using umbrella branding, as in case (U,U), we obtain Lemma 7, where thresholds $\lambda_{sc}^{(1)}$, $\lambda_{sc}^{(2)}$, and $\lambda_{sc}^{(3)}$ are given, respectively, by (48), (49), and (50) in Appendix B.2.

LEMMA 7. *Suppose markets s and f are fully covered. Then:*

(i) If one brand does not extend, then it is always beneficial for the other brand to extend via umbrella branding (i.e., $\Pi^A(U,NL) > \Pi^A(NL,NL)$ and $\Pi^B(NL,U) > \Pi^B(NL,NL)$) if, and only if, $\lambda_s < \lambda_{sc}^{(1)}$.

(i) If one brand extends via umbrella branding, then it is always beneficial for the other brand to extend via umbrella branding (i.e., $\Pi^A(U,U) > \Pi^A(NL,U)$ and $\Pi^B(U,U) > \Pi^B(U,NL)$) if, and only if, $\lambda_s \in \left(\lambda_{sc}^{(2)}, \lambda_{sc}^{(3)}\right)$.

When both brands do not extend, Lemma 7(i) reveals that one of the brands is better off deviating and extending via umbrella branding only if the snobs' negative popularity effect is sufficiently low (i.e., $\lambda_s <$

$\lambda_{sc}^{(1)}$). This result is due to the fact that, when the snobs' negative popularity effect is sufficiently low, the revenue gain from extending in market f outweighs the profit loss from market s.

Next, when both brands extend via umbrella branding, Lemma 7(ii) shows that one of the brands is better off deviating and not extending when the snobs' negative popularity effect is either sufficiently high or sufficiently low. In the former case (i.e., when $\lambda_s \geq \lambda_{sc}^{(3)}$), the brand that deviates by not extending will become the monopoly in market s (see case 3 above) and earns a higher profit by focusing on market s. In the latter case (i.e., when $\lambda_s \leq \lambda_{sc}^{(2)}$ which occurs when the followers' positive popularity effect λ_f is relative high), it can be shown that the indirect effect of umbrella branding will intensify competition in market s as explained earlier. Consequently, one of the brands is better off not extending by focusing on the sales in market s and letting the other brand to extend and sell more in market f.

7.3. Equilibrium brand extension strategy with the umbrella branding

By comparing the brands' profits under different umbrella branding strategies (i.e., (NL, NL), (U, NL), and (U, U)) and by applying Lemma 7, we can characterize the equilibrium brand extension strategies in the following proposition.

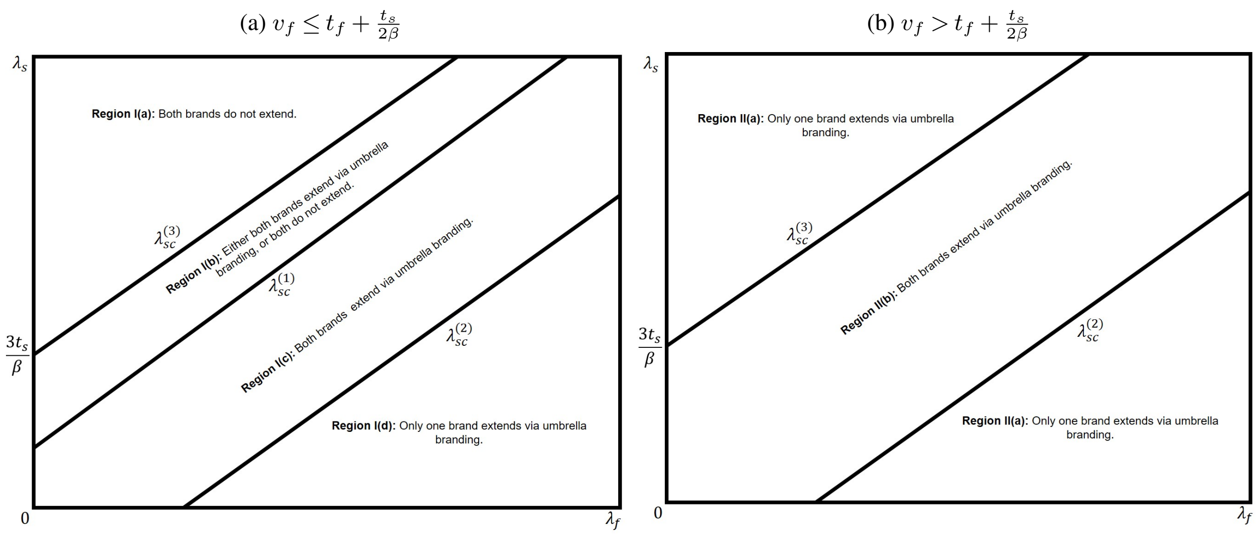
PROPOSITION 4. *Suppose both markets s and f are fully covered. Then, under the umbrella branding, the equilibrium brand extension strategy to be adopted by both brands can be characterized as follows:*

I. When the base valuation of the product in market f is low so that $v_f \leq t_f + \frac{t_s}{2\beta}$,
   - *(a) both brands do not extend if $\lambda_s \geq \lambda_{sc}^{(3)}$;*
   - *(b) both brands either extend or do not extend (i.e., two equilibria exist) if $\lambda_{sk}^{(1)} < \lambda_s < \lambda_{sk}^{(3)}$;*
   - *(c) both brands extend if $\lambda_{sc}^{(2)} < \lambda_s < \lambda_{sc}^{(1)}$;*
   - *(d) only one brand extends if $\lambda_s \leq \lambda_{sc}^{(2)}$.*

II. When the base valuation of the product in market f is high so that $v_f > t_f + \frac{t_s}{2\beta}$,
   - *(a) only one brand extends if $\lambda_s \leq \lambda_{sc}^{(2)}$, or if $\lambda_s \geq \lambda_{sc}^{(3)}$;*
   - *(b) both brands extend if $\lambda_{sc}^{(2)} < \lambda_s < \lambda_{sc}^{(3)}$.*

Observe that Proposition 4 possesses a similar structure as Proposition 2. Also, observe from Appendix B.2 that the thresholds $\lambda_{sc}^{(1)}$, $\lambda_{sc}^{(2)}$ and $\lambda_{sc}^{(3)}$ are linear functions of λ_f. These two observations enable us to illustrate Proposition 4(I) for the case $v_f \leq t_f + \frac{t_s}{2\beta}$ in Figure 2a. Similar to the case where brands use fixed-fee contracts to license (see Figure 1a), there are four distinct regions where brands prefer different umbrella branding strategies. Comparing Figures 1a and 2a, we observe that, when snobs' sensitivity to brand popularity is sufficiently high (i.e., $\lambda_s \geq 3t_s/\beta$), no brand extends through licensing under fixed-fee contracts; on the other hand, depending on followers' desire to adopt the brand (λ_f), there might be a brand that extends under the umbrella branding. Brands have different objectives with the fixed-fee contract and umbrella branding strategy. Specifically, given the fixed lump-sum payment under a fixed-fee contract, brands aim to maximize only their profits from market s, whereas under the umbrella branding strategy, they aim to maximize total profits from both markets. For $\lambda_s \geq 3t_s/\beta$, the negative popularity effect significantly reduces brands' profits from market s. Therefore, for lower λ_f values when $\lambda_s \geq 3t_s/\beta$ (region I(a)), no brand extends under the umbrella branding strategy, and as we formalize in Corollary 4, both brands would earn more profits if both could commit to extend in such cases (by Lemma 6(i)), but they face a prisoner's dilemma and do not extend. However, for higher λ_f values, it becomes more attractive for brands to extend, and at least one of the brands extends under the umbrella branding strategy (e.g., regions I(b), I(c) and I(d) for $\lambda_s \geq 3t_s/\beta$).

Figure 2 Brands' Equilibrium Extension Strategies under Umbrella Branding



COROLLARY 4. *Suppose markets s and f are fully covered. In equilibrium under the umbrella branding, when $v_f \leq t_f + \frac{t_s}{2\beta}$, if $\lambda_s \geq \lambda_{sc}^{(3)}$, brands face prisoner's dilemma and both do not extend, even though each brand would be better off if both brands extended via umbrella branding.*

It is interesting to note from Figure 2a that, unlike the case when both brands license under fixed-fee contracts as in region I(d) as shown in Figure 1a, only one brand extends under the umbrella branding for sufficiently high λ_f and sufficiently low λ_s values (i.e., region I(d) for $\lambda_s < 3t_s/\beta$). In such cases, the competition between brands becomes very intense due to high λ_f when both brands use umbrella branding. Therefore, one of the brands is always better off from not extending. Finally, extending in market f becomes more attractive when followers' base valuation is high enough ($v_f > t_f + \frac{t_s}{2\beta}$). Hence, as Figure 2b illustrates, it is always beneficial for at least one brand to extend in market f in such cases.

8. Concluding Remarks

Over the last 30 years, many luxury brands have licensed their brand names to licensees so that they can extend their product offerings in new product categories in a cost-effective and time-efficient manner [License Global, 2004]. While licensing can enable a luxury brand to capture sales from aspirational consumers (or followers), it can backfire and make the brand less attractive for the exclusivity-seeking consumers (or snobs) who purchase brands' own high-quality products. To examine these two countervailing forces associated with licensing, we have developed a game-theoretic model to investigate how conspicuous consumption and competition affect licensing strategies of luxury brands.

Fixed-fee licensing. Through our analysis of fixed-fee licensing contracts, we have obtained the following results. First, in the monopoly setting, we have shown that the brand should not license when the snobs' negative popularity effect is sufficiently high because licensing will cause the brand to suffer from a significant drop in the profit generated by the snobs. However, in a duopoly, we have shown that licensing is always beneficial for both brands, even when the brands license their brand names for free. The intuition of this result is primarily driven by the indirect effect of licensing that softens competition between brands. Second, we have characterized the equilibrium licensing strategy adopted by both brands and discovered two key results. First, when the snobs' negative popularity effect is above a certain threshold, each brand would have earned more if they could both commit to licensing. However, in the absence of such a commitment, we have showed that both brands face a prisoner's dilemma and do not license in equilibrium. Second, when the snobs' negative popularity effect is below a certain threshold, we have mapped out the conditions under which both brands, or only one brand, would license in equilibrium.

Royalty licensing and umbrella branding. We have also extended our analysis to examine the issue of royalty licensing contracts and umbrella branding strategies. Unlike fixed-fee licensing contracts, our analysis revealed that royalty licensing contracts intensify competition between brands because both brands need to compete for the snobs' demand so that they can attract more followers to purchase the licensed products. Therefore, when the followers' positive popularity effect is strong, competition is too strong under royalty licensing contracts, and fixed-fee contracts dominate royalty contracts. Akin to the royaty contract, we have discovered that umbrella branding intensifies competition so that fixed-fee licensing contracts dominates umbrella branding when the followers' positive popularity effect is strong. This result implies that licensing in a decentralized system can be more efficient than umbrella branding in a centralized system. Therefore, in a setting where the consumers' popularity effect plays an important role, a decentralized system can be more efficient. Hence, our result is in contrast to the supply-chain contract literature [Pasternack, 1985; Taylor, 2002; Cachon and Lariviere, 2005]) arguing that a centralized system is always more efficient.

Future research. There are several directions for future research. First, we have assumed that brands (and licensees) are horizontally differentiated. Future work can examine settings where competing firms (brands and/or licensees) are vertically differentiated. Second, for tractability, we have assumed that the snob and follower markets are completely separate so that snobs purchase only luxury brands' high-quality products while followers purchase only licensees' low-quality products. Future research can study alternative settings where snobs and/or followers may purchase both luxury brands' products and licensed products. Lastly, we have assumed that consumers in one market are only sensitive towards brand popularity across the other market. However, our model can be generalized by considering cases where consumers are sensitive towards brand popularity within a market.

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Appendix

A. Partially-Covered Snob Market (Market s)

In this appendix, we will analyse the monopoly setting under the fixed-fee contract when market s is partially covered. Suppose brand A located at 0 is the monopoly that considers licensing its brand name to licensee a. We assume the same sequence events as in Section 4.

First, as a benchmark, let us consider the case where brand A does not license (NL), i.e., $D_f^a = 0$ and $T^A = 0$. The net utility of snob located at θ from purchasing brand A's product is equal to $U^A(\theta) = v_s - t_s\theta_s - p^A$. Thus, the marginal snob is given by $\theta_s = \left(v_s - p^A\right)/t_s$. Using rational expectations (i.e., $D_s^A = D_s^{A(e)} = \theta_s$), we solve brand A's problem (i.e., $\max_{p^A} \Pi^A(p^A) = (p^A - c)D_s^A$), and we obtain $p^A = (v_s + c)/2$. Substituting the price p^A to retrieve θ_s, we have $\theta_s = (v_s - c)/2t_s$. This implies that market s is partially-covered when brand A does not license if $v_s - c < 2t_s$. To make a fair comparison between the case where brand A does not license and licenses by using fixed-fee contract when market s is partially-covered, we will assume that $v_s - c < 2t_s$ throughout this appendix. Finally substituting $p^A = (v_s + c)/2$, $D_s^A = \theta_s$, and $T^A = 0$ in (3), brand A's profit is given by:

$$\Pi^A(NL) = \frac{(v_s - c)^2}{4t_s}. \tag{31}$$

Next, we will first analyze the case where market f is fully covered and then analyze the case where market f is partially covered.

A.1. Fully-covered followers' market (market f)

Consider the case where market f is fully covered so that licensee a sets its price equal to $p^a = v_f + \lambda_f D_s^A - t_f$ to cover market f, i.e., $\theta_f = 1$. Using rational expectations (i.e., $D_f^{a(e)} = 1$), marginal snob is given by:

$$\theta_s = \frac{v_s - p^A - \beta\lambda_s}{t_s}.$$

Then, by rational expectations (i.e., $D_s^A = D_s^{A(e)} = \theta_s$) and substituting $T^A = k^A$ into (3), we obtain brand A's profit. By using first-order conditions, brand A's and licensee a's equilibrium prices are given, respectively, by:

$$p^A = \frac{v_s + c - \beta\lambda_s}{2},\ p^a = v_f + \lambda_f \frac{v_s - c - \beta\lambda_s}{2t_s} - t_f.$$

Substituting p^A to retrieve θ_s, we obtain

$$\theta_s = \frac{v_s - c - \beta\lambda_s}{2t_s}.$$

Note that combined with our assumption that $v_s - c < 2t_s$ (to ensure market s is partially-covered in case of no licensing, see above), market s is partially covered (i.e., $\theta_s \in (0,1)$) if $\lambda_s < (v_s - c)/\beta$. Therefore, we will focus on $\lambda_s < (v_s - c)/\beta$ to ensure market s is partially covered when brand A uses fixed-fee contract.

Brand A sets its fixed fee equal to licensee a's revenue to extract licensee a's revenue and the fixed lump-sum payment is given by:

$$k^A = \left(v_f + \lambda_f \frac{v_s - c - \beta\lambda_s}{2t_s} - t_f\right)\beta.$$

Notice from above that brand A's revenue from licensing is always higher than its profit in the case of no licensing, i.e., $k^A > \Pi^A(NL)$, if followers' base valuation is sufficiently high (i.e., $v_f \geq t_f + \frac{(v_s - c)^2}{4\beta t_s}$). In such cases, brand A always uses fixed-fee contract to license. Next, we will focus on cases where market f is fully covered and followers' base valuation is sufficiently small, i.e., $v_f < t_f + \frac{(v_s - c)^2}{4\beta t_s}$.

Substituting p^A, $D_s^A = \theta_s$, and $T^A = k^A$ into (3), brand A's profit under fixed-fee contract is equal to

$$\Pi^A(F) = \frac{(v_s - c - \beta\lambda_s)^2}{4t_s} + \left(v_f + \lambda_f \frac{v_s - c - \beta\lambda_s}{2t_s} - t_f\right)\beta. \quad (32)$$

Comparing brand A's profit under the fixed-fee contract above against that in the case of no licensing in (31), we obtain Lemma 8, where the threshold $\hat{\lambda}_{sk}^{(1)} \in (0, \frac{v_s - c}{\beta})$ satisfies

$$\hat{\lambda}_{sk}^{(1)} = \frac{v_s - c}{\beta} + \lambda_f - \sqrt{\lambda_f^2 + \left(\frac{v_s - c}{\beta}\right)^2 - 4t_s \frac{v_f - t_f}{\beta}}. \quad (33)$$

We present proofs of all results in this appendix in Appendix E. Lemma 8 confirms our result in Proposition 1 and shows that when market s is partially covered, brand A does not license if snobs' sensitivity to brand popularity is sufficiently high.

LEMMA 8. *Suppose that market s is partially-covered and market f is fully-covered. When followers' base valuation is sufficiently small (i.e., $v_f < t_f + \frac{(v_s - c)^2}{4\beta t_s}$), brand A licenses its brand name to licensee a by using fixed-fee contract if, and only if, $\lambda_s < \hat{\lambda}_{sk}^{(1)}$.*

A.2. Partially-covered followers' market (market f)

Consider the case where market f is also partially covered, i.e., $\theta_f < 1$. Using rational expectations, we obtain marginal snob and followers, respectively, as follows:

$$\theta_s = \frac{t_f\left(v_s - p^A\right) + \beta\lambda_s\left(p^a - v_f\right)}{t_f t_s + \beta\lambda_f\lambda_s},$$
$$\theta_f = \frac{\lambda_f\left(v_s - p^A\right) - \left(p^a - v_f\right)t_s}{t_f t_s + \beta\lambda_f\lambda_s}.$$

By using rational expectations and substituting $D_s^A = \theta_s$, $D_f^a = \theta_f$ and $T^A = k^A$ into (3) and (4), we obtain profits of brand A and licensee a. Then by first-order conditions, we obtain equilibrium prices of brand A and licensee a as follows:

$$p^A = \frac{2t_f t_s\left(v_s + c\right) + \beta\lambda_s\left(\lambda_f v_s - v_f t_s\right)}{4t_f t_s + \beta\lambda_f\lambda_s},$$
$$p^a = \frac{\lambda_f t_f\left(v_s - c\right) + v_f\left(2t_f t_s + \beta\lambda_f\lambda_s\right)}{4t_f t_s + \beta\lambda_f\lambda_s}.$$

Substituting p^A and p^a back to retrieve θ_s and θ_f, we obtain

$$\theta_s = t_f\frac{\left(2t_f t_s + \beta\lambda_f\lambda_s\right)\left(v_s - c\right) - \beta\lambda_s v_f t_s}{\left(t_f t_s + \beta\lambda_f\lambda_s\right)\left(4t_f t_s + \beta\lambda_f\lambda_s\right)}, \quad (34)$$
$$\theta_f = t_s\frac{\lambda_f t_f\left(v_s - c\right) + v_f\left(2t_f t_s + \beta\lambda_f\lambda_s\right)}{\left(t_f t_s + \beta\lambda_f\lambda_s\right)\left(4t_f t_s + \beta\lambda_f\lambda_s\right)}. \quad (35)$$

Under certain sufficient conditions, market s and f are partially covered (i.e., $\theta_s, \theta_f \in (0,1)$), see Lemma 9 below.

LEMMA 9. *If $(v_s - c) \leq \min\left(\beta\lambda_s, 2t_s\right)$ and $\lambda_f > \frac{v_f t_s}{v_s - c}$, market s and f are partially-covered, i.e., $\theta_s \in (0,1)$ and $\theta_f \in (0,1)$.*

Brand A sets its fixed lump-sum payment to extract all revenues of licensee a, i.e., $k^A = p^a\beta\theta_f$. Thus the fixed lump-sum payment of brand A is given by:

$$k^A = \beta t_s\frac{\left(\lambda_f t_f\left(v_s - c\right) + v_f\left(2t_f t_s + \beta\lambda_f\lambda_s\right)\right)^2}{\left(t_f t_s + \beta\lambda_f\lambda_s\right)\left(4t_f t_s + \beta\lambda_f\lambda_s\right)^2}.$$

Then substituting $D_s^A = \theta_s$, k^A and p^A into (3), we obtain brand A's equilibrium profit as follows:

$$\Pi^A = t_f\frac{\left(\left(2t_f t_s + \beta\lambda_f\lambda_s\right)\left(v_s - c\right) - \beta\lambda_s v_f t_s\right)^2}{\left(t_f t_s + \beta\lambda_f\lambda_s\right)\left(4t_f t_s + \beta\lambda_f\lambda_s\right)^2} + \beta t_s\frac{\left(\lambda_f t_f\left(v_s - c\right) + v_f\left(2t_f t_s + \beta\lambda_f\lambda_s\right)\right)^2}{\left(t_f t_s + \beta\lambda_f\lambda_s\right)\left(4t_f t_s + \beta\lambda_f\lambda_s\right)^2}. \quad (36)$$

It is analytically very challenging to compare brand A's profit in (36) against that in the case of no licensing in (31). Therefore, we resort to numerical examples. Figure 3 illustrates $\Pi^A(F) - \Pi^A(NL)$ as a function of λ_s for parameter values that are selected according to Lemma 9 to ensure market s and f are partially covered. Figure 3 confirms Proposition 4 and shows that even when market s and f are partially covered, brand A is better off from licensing via fixed-fee contract only if snobs' sensitivity to brand popularity is less than a certain threshold.

B. Definitions of Threshold λ_s and λ_f Values in Sections 6 and 7

In this appendix, we define the threshold λ_s and λ_f values that we use in the paper.

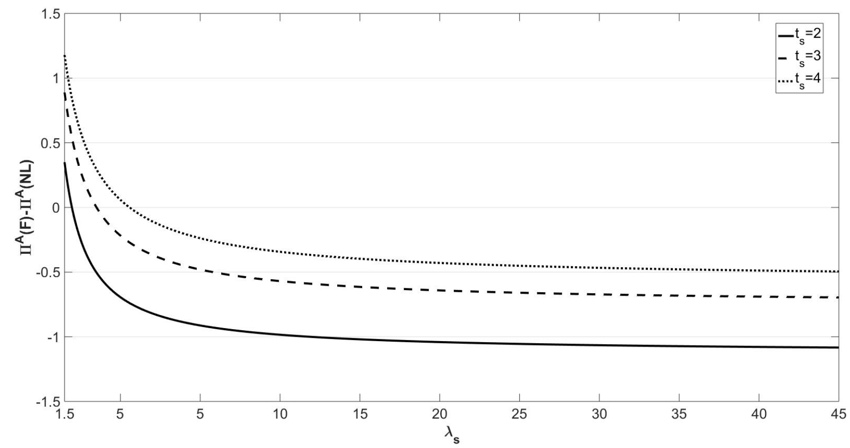

Figure 3 **The increase in brand A's profit due to licensing via fixed-fee contract (F) relative to the case of no licensing (NL) (i.e., $\Pi^{\mathbf{A}}(\mathbf{F})-\Pi^{\mathbf{A}}(\mathbf{NL})$) when $\mathbf{v_s=4, v_f=2, c=1, t_f=1,}$ and $\beta=\mathbf{2}$. The parameter values in the figure are determined in line with Lemma 9 to ensure market s and f are partially-covered.**

B.1. Thresholds in Section 6

Let us define functions $g_j\left(\lambda_f\right)$ and $h_j\left(\lambda_f\right)$ for $j=1,2$ as follows:

$$g_1\left(\lambda_f\right) = 2\beta\lambda_f^3-4\beta\lambda_f^2t_f+4\beta\lambda_f t_f^2+\lambda_f t_f t_s+2t_f^2t_s \tag{37}$$

$$g_2\left(\lambda_f\right) = 2\beta^2\lambda_f^3+4\beta\lambda_f^2t_s+2\lambda_f t_s^2-\beta\lambda_f t_f t_s-2t_f t_s^2, \tag{38}$$

$$h_1\left(\lambda_f\right) = 3t_f t_s\left(3t_f-\lambda_f\right)\left(\lambda_f-t_f\right), \tag{39}$$

$$h_2\left(\lambda_f\right) = t_s\left(3t_s-2\beta t_f\right)\lambda_f^2-12t_f t_s^2\lambda_f+6t_f^2t_s^2. \tag{40}$$

We define threshold λ_s values in Section 6 as follows:

$$\lambda_{sr}^{(1)} = \frac{3t_s\left(\lambda_f-3t_f\right)}{2\left(t_s+2\beta\lambda_f\right)}, \tag{41}$$

$$\lambda_{sr}^{(2)} = \frac{g_1\left(\lambda_f\right)}{h_1\left(\lambda_f\right)}, \tag{42}$$

$$\lambda_{sr}^{(3)} = \frac{g_2\left(\lambda_f\right)}{h_2\left(\lambda_f\right)}, \tag{43}$$

$$\lambda_{sr}^{(5)}=\frac{3t_s}{2\beta}\left(\frac{\lambda_f}{t_f}+\frac{4\lambda_f\beta\left(\lambda_f-3t_f\right)\left(\lambda_f-t_f\right)}{3t_f^2t_s+2\beta t_f\lambda_f^2} +\sqrt{\left(\frac{\lambda_f}{t_f}+\frac{4\lambda_f\beta\left(\lambda_f-3t_f\right)\left(\lambda_f-t_f\right)}{3t_f^2t_s+2\beta t_f\lambda_f^2}\right)^2+3\frac{4\beta\left(\lambda_f-3t_f\right)\left(\lambda_f-t_f\right)}{3t_f t_s+2\beta\lambda_f^2}}\right). \tag{44}$$

Lastly, we let thresholds $\lambda_{fr}^{(1)}$ and $\lambda_{fr}^{(2)}$ be unique $\lambda_f\in(0,t_f)$ values that, respectively, satisfy:

$$g_2(\lambda_{fr}^{(1)}) = 0, \tag{45}$$

$$h_2(\lambda_{fr}^{(2)}) = 0. \tag{46}$$

B.2. Thresholds in Section 7

We define threshold λ_f and λ_s values that we use in Section 7 as follows:

$$\lambda_{fc}^{(1)} = \frac{\lambda_s t_f t_s}{\beta\lambda_s t_f + \lambda_s t_s + t_f t_s}, \tag{47}$$

$$\lambda_{sc}^{(1)} = \begin{cases} \lambda_f + \frac{3t_s}{\beta} - \frac{3t_s}{\beta}\sqrt{1-2\beta\frac{v_f - t_f}{t_s}}, & \text{if } v_f < t_f + \frac{t_s}{2\beta}, \\ \infty, & \text{if } v_f \geq t_f + \frac{t_s}{2\beta}, \end{cases} \tag{48}$$

$$\lambda_{sc}^{(2)} = \begin{cases} \left(\lambda_f + \frac{3t_s}{2\beta} - \frac{3t_s}{2\beta}\sqrt{1+4\beta\frac{t_f}{t_s}}\right)^+, & \text{if } t_f < \frac{t_s}{2\beta}, \\ (\lambda_f - t_f - t_s/\beta)^+, & \text{if } t_f \geq \frac{t_s}{2\beta}, \end{cases} \tag{49}$$

$$\lambda_{sc}^{(3)} = \lambda_f + \frac{3t_s}{2\beta}. \tag{50}$$

C. Royalty Contract in the Monopoly Setting

In this appendix, we study the royalty contract in the monopoly setting. Consider the same sequence of events as in Section 4, and assume that brand A sells as a monopoly in market s and licenses its brand to licensee a who sells the licensed product in market f via a royalty contract. For each unit sold in market f, brand A charges r^A to licensee a and hence the transfer payment between brand A and licensee a is equal to $T^A = r^A\beta D_f^a$.

In market f, the marginal follower is located at $\theta_f = \min\left\{\frac{v_f+\lambda_f-p^a}{t_f}, 1\right\}$ as in (6). By noting that $D_f^a = \theta_f$, brand A sets its price equal to $p^A = v_s - t_s - \lambda_s\beta\theta_f$ to ensure that market s is fully covered (i.e., $D_s^A = 1$). By substituting $D_f^a = \theta_f$ and $T^A = r^A\beta D_f^a$ in (4), we obtain licensee a's profit and then determine the optimal price p^a, where:

$$p^a = \begin{cases} \frac{v_f+\lambda_f+r^A}{2}, & \text{if } r^A \leq v_f + \lambda_f \\ v_f + \lambda_f, & \text{if } r^A > v_f + \lambda_f. \end{cases}$$

Through substitution, $\theta_f = (v_f + \lambda_f - p^a)/2t_f$. Then by using $p^A = v_s - t_s - \lambda_s\beta\theta_f$, $D_s^A = 1$ and $T^A = r^A\beta\theta_f$, we use (3) to obtain brand A's profit and then we determine the optimal royalty fee r^A, where:

$$r^A = \begin{cases} v_f + \lambda_f - 2t_f, & \text{if } \lambda_s \leq v_f + \lambda_f - 4t_f, \\ \frac{v_f+\lambda_f+\lambda_s}{2}, & \text{if } v_f + \lambda_f - 4t_f < \lambda_s < v_f + \lambda_f, \\ v_f + \lambda_f, & \text{if } \lambda_s \geq v_f + \lambda_f. \end{cases}$$

We can then retrieve the optimal prices of brand A and licensee a as follows:

$$p^A = v_s - t_s - \frac{\lambda_s\beta}{2t_f}\left(v_f + \lambda_f - r^A\right),\ p^a = \frac{v_f + \lambda_f + r^A}{2}.$$

Observe that licensee a's price depends on the royalty fee r^A. This observation implies that brand A can directly control licensee a's price (and its demand) under the royalty contract as follows. First, when the snobs' 'negative popularity effect' is significantly high (i.e., $\lambda_s \geq v_f + \lambda_f$), brand A will set a very high royalty fee $r^A = v_f + \lambda_f$ so that licensee a will need to charge a very high price $p^a = v_f + \lambda_f$. In this case, the corresponding $\theta_f = 0$ so that no one will purchase the licensed product in market f. This is equivalent to the case of no licensing. Second, when the snobs' negative popularity effect is low, i.e., $\lambda_s \leq v_f + \lambda_f - 4t_f$, brand A can afford to set a lower royalty fee r^A so that licensee a can cover the entire market f. In doing so, brand A can collect a higher licensing revenue without alienating the snobs in market s. Third, for intermediate values of λ_s, brand A will set its royalty fee r^A so that market f is partially covered. With a lower demand in market f, the snobs are less alienated, which allows brand A to charge a higher price for its own product because p^A is increasing in r^A.

Lastly, different from the fixed-fee contract, licensee a can obtain a positive profit in equilibrium under the royalty contract (i.e., $\Pi^a > 0$) since $r^A < p^a$ when $\lambda_s < v_f + \lambda_f$. Unlike the fixed-fee contract with which brand A extracts all revenues of licensee a (i.e., $\Pi^a = 0$) because it has no control on its licensee's price (and demand), the royalty contract enables brand A to control licensee a's price (and demand). In return of this control, brand A gives up on some licensing revenues. Therefore, brand A will behave differently under fixed-fee and royalty contracts, which we shall examine in more details later.

Through substitution, $\theta_f = (v_f + \lambda_f - r^A)/2t_f$. Combine this and substituting $p^A = v_s - t_s - \lambda_s \beta \theta_f$, $D_s^A = 1$ and $T^A = r^A \beta \theta_f$ into (3), we retrieve the optimal profit of brand A as:

$$\Pi^A(R) = v_s - t_s - c + \frac{\beta}{2t_f}(r^A - \lambda_s)\left(v_f + \lambda_f - r^A\right). \tag{51}$$

Recall from (5) that, without licensing, brand A's profit $\Pi^A(NL) = v_s - t_s - c$. Also, with licensing, its profit under the fixed-fee contract is given in (8). By comparing brand A's profit under the royalty contract $\Pi^A(R)$ in (51) against these two profits, we get:

PROPOSITION 5. *Suppose market s is fully covered. Then:*
(i) Brand A licenses under the royalty contract, i.e., $\Pi^A(R) > \Pi^A(NL)$, if, and only, if $\lambda_s < v_f + \lambda_f$.
(ii) Brand A prefers the royalty contract over the fixed-fee contract, i.e., $\Pi^A(R) > \Pi^A(F)$, if, and only if: (1) $\lambda_s > \left(\sqrt{2}-1\right)(v_f + \lambda_f)$ when $\lambda_f < 2t_f - v_f$; or (2) $\lambda_s > v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f$ when $\lambda_f \geq 2t_f - v_f$.

Proofs of the results in this appendix are deferred to Appendix E. Comparing Proposition 1 and Proposition 5(i), we observe that brand A licenses in a wider range of λ_s under the royalty contract. Through the royalty fee, brand A can control its licensee's sales and curb the negative impact of licensing on snobs, which enables him to license for higher λ_s values.

Proposition 5(ii) shows that brand A prefers the royalty contract when the snobs' negative popularity effect λ_s is sufficiently high. While brand A can extract the entire surplus of licensee a under the fixed-fee contract, the royalty contract enables the brand to control its licensee's price (and demand). When λ_s is high, it is important for brand A to strike the balance between the sales of its own luxury goods in market s and the royalties to be collected from licensee a (that is based on the sales of the licensed product in market f). Consequently, by leveraging this control capability for the case when λ_s is high, the royalty contract dominates the fixed-fee contract when a monopoly brand licenses its name to a licensee. This result explains why royalty contracts are commonly observed in highly conspicuous markets [Centre for Fashion Enterprise, 2012]. Proposition 5 also complements the literature by showing that conspicuous consumer behavior can be another rationale behind royalty contracts in addition to uncertain demand [Bousquet et al., 1998], asymmetric information [Beggs, 1992; Gallini and Wright, 1990; Choi, 2001], and competition [Wang, 1998; Poddar and Sinha, 2002].

D. Umbrella Branding in the Monopoly Setting

We analyse the umbrella branding strategy in the monopoly setting in this appendix. We assume that brand A is located at 0 of the Hotelling line and operates as a monopoly in markets s and f. The sequence of events can be described as follows. First, brand A decides whether to extend in market f by launching the low-quality product. Then, if it decides to extend in market f, it simultaneously determines prices in market s and f, i.e., p^A and p^a. If brand A decides to not extend, it is a monopoly only in market s and thus decides p^A as in (5) in §4.1. Finally consumers in market s and f decide whether to purchase or not. Following a similar procedure as in §4.2 and using (25), we determine equilibrium prices of brand A as follows:

$$p^A = \begin{cases} v_s - t_s - \lambda_s \beta, & \text{if } \lambda_s \le v_f + \lambda_f - 2t_f, \\ v_s - t_s - \lambda_s \beta \frac{v_f + \lambda_f - \lambda_s}{2t_f}, & \text{if } v_f + \lambda_f - 2t_f < \lambda_s < v_f + \lambda_f, \\ v_s - t_s, & \text{if } \lambda_s \ge v_f + \lambda_f, \end{cases}$$

$$p^a = \begin{cases} v_f + \lambda_f - t_f & \text{if } \lambda_s \le v_f + \lambda_f - 2t_f, \\ \frac{v_f + \lambda_f + \lambda_s}{2} & \text{if } v_f + \lambda_f - 2t_s < \lambda_s < v_f + \lambda_f, \\ v_f + \lambda_f & \text{if } \lambda_s \ge v_f + \lambda_f. \end{cases}$$

Using above prices and market s being covered (i.e., $D_s^A = 1$), we determine marginal follower θ_f by (6). Then substituting prices, $D_s^A = 1$ and $D_f^a = \theta_f$ into (25), brand A's total profit obtained from both markets under umbrella branding is given by:

$$\Pi^A(U) = \begin{cases} v_s - t_s - c - \lambda_s \beta + \beta\left(v_f + \lambda_f - t_f\right), & \text{if } \lambda_s \le v_f + \lambda_f - 2t_f, \\ v_s - t_s - c + \beta \frac{\left(v_f + \lambda_f - \lambda_s\right)^2}{4t_f}, & \text{if } v_f + \lambda_f - 2t_f < \lambda_s < v_f + \lambda_f, \\ v_s - t_s - c, & \text{if } \lambda_s \ge v_f + \lambda_f. \end{cases} \tag{52}$$

For $\lambda_s \ge v_f + \lambda_f$, the negative popularity effect is very high so that brand A sets a very high price and no followers purchase in market f (i.e., $\theta_f = 0$ by (6)). Thus, in line with Amaldoss and Jain [2015], extending in market f via umbrella branding is optimal only when the snobs' negative popularity effect λ_s is sufficiently small. This is formalized in Proposition 6(i). We omit the proof of Proposition 6.

PROPOSITION 6. *Suppose market s is fully covered.*

(i) Brand A extends in market f if, and only if, $\lambda_s < v_f + \lambda_f$.

(ii) Relative to those under the umbrella branding, equilibrium prices in market s and f are different and brand A obtains lower profits under both fixed-fee and royalty contracts if $\lambda_s < v_f + \lambda_f$.

Brand A aims to maximize the total profits from both markets under the umbrella branding (centralized system). Hence, the umbrella branding is actually the first best in the monopoly setting. Comparing the monopoly prices under fixed-fee and royalty contracts (Sections 4 and Appendix C) against the monopoly prices as stated above, we show that fixed-fee and royalty contracts cannot attain the same profit under the umbrella branding and both contracts are not fully-efficient due to decentralization. This is formalized in Proposition 6(ii). Under the fixed-fee contract, the brand has no control on its licensee's price (and demand). Licensee a pays the fixed-fee at the beginning and charges a high price p^a to maximize its profit from market f. However, when the brand owns the product under umbrella branding, the brand will choose a lower price and sell more. This decreases revenues of brand A from market s and hence total profits are lower compared to those under the umbrella branding. Brand A can affect licensee a's price (and demand) under the royalty contract. However, since it is concerned about the negative popularity effect, it charges a high royalty fee to limit its licensee's sales. In return, licensee a charges a high price and less followers purchase the product.

E. Proofs

Proof of Proposition 1: Let us define $\check{\lambda}_{sk}^{(1)} \in (0, v_f + \lambda_f)$ as follows:

$$\check{\lambda}_{sk}^{(1)} = \begin{cases} \frac{v_f+\lambda_f}{2}, & if \ \lambda_f < 2t_f - v_f, \\ v_f + \lambda_f - t_f, & if \ \lambda_f \geq 2t_f - v_f. \end{cases} \tag{53}$$

Comparing (5) and (8), the brand A licenses under fixed-fee contract, if and only if, $\lambda_s < \check{\lambda}_{sk}^{(1)} \in (0, v_f + \lambda_f)$. □

Proof of Lemma 2: We will prove each part of the lemma separately.

(i) In this part, we characterize the benefit to a brand from licensing via fixed-fee contract when the other brand does not license to determine cases. Without loss of generality, assume that brand A licenses while brand B does not license. We let $\Delta(F, NL) = \Pi^A(F, NL) - \Pi^A(NL, NL)$ be the benefit of using fixed-fee contract for brand A when the other brand does not license. By symmetry (i.e., $\Pi^B(NL, F) = \Pi^A(F, NL)$ and $\Pi^A(NL, F) = \Pi^B(F, NL)$), the benefit to brand B from licensing via fixed-fee contract when brand A does not license is also equal to $\Delta(F, NL)$. By (9) and (17) , we have

$$\Delta(F, NL) = 2t_s \left(\left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s} \right)^+ \right)^2 + \left(v_f + \lambda_f \left(\frac{1}{2} - \frac{\beta\lambda_s}{6t_s} \right)^+ - t_f \right) \beta - \frac{t_s}{2}.$$

Consider two cases: (a) $v_f \leq t_f + t_s/2\beta$, and (b) $v_f > t_f + t_s/2\beta$.

Case (ia): In this case, $\lim_{\lambda_s \downarrow 0} \Delta(F, NL) = \frac{1}{2}\beta(\lambda_f + 2(v_f - t_f)) > 0$, $\lim_{\lambda_s \uparrow 3t_s/\beta} \Delta(F, NL) < 0$ (since $v_f < t_f + t_s/2\beta$) and $\Delta(F, NL)$ is strictly decreasing in λ_s. This indicates that there exists a unique $\lambda_{sk}^{(1)} \in (0, 3t_s/\beta)$ such that for $\lambda_s < \lambda_{sk}^{(1)}$, both brands are better off when only one brand licenses via fixed-fee contract. In all other cases ($\lambda_s \geq \lambda_{sk}^{(1)}$), the brand that licenses (brand A) is better off if she does not license. Solving for λ_s such that $\Delta(F, NL) = 0$, we obtain $\lambda_{sk}^{(1)}$ in (19) for $v_f \leq t_f + t_s/2\beta$.

Case (ib): In this case, $\Delta(F, NL) > 0$ for all λ_s, i.e., $\lambda_s < \lambda_{sk}^{(1)} = \infty$. Thus, both brands are better off when only one of them licenses via fixed-fee contract compare to the case where no brand licenses.

(ii) In this part, we characterize the benefit to a brand from licensing via fixed-fee contract when the other brand also licenses via fixed-fee contract. To that end, we let $\Delta(F, F)$ denote the benefit to a brand from licensing via fixed-fee contract when the other brand uses fixed-fee contract. Note that $\Pi^A(F, F) = \Pi^B(F, F)$ by (14) and $\Pi^A(NL, F) = \Pi^B(F, NL)$ by (18). Therefore, the benefit from licensing via fixed-fee contract when the other brand uses fixed-fee contract is the same for brand A and B. First, consider, $\lambda_s < 3t_s/\beta$. In this case, by (14) and (18),

$$\Delta(F, F) = \frac{t_s}{2} + \frac{\beta\lambda_f\lambda_s}{2t_f} + \frac{\beta t_f}{2} + \frac{\beta^2\lambda_f\lambda_s}{2t_s} - 2t_s \left(\frac{1}{2} + \frac{\beta\lambda_s}{6t_s} \right)^2.$$

Note that $\lim_{\lambda_s \downarrow 0} \Delta(F, F) > 0$, $\lim_{\lambda_s \uparrow \infty} \Delta(F, F) < 0$, and $\Delta(F, F)$ is a concave quadratic function of λ_s. This indicates that there exists $\lambda_{sk}^{(2)} \in (0, \infty)$ such that $\Delta(F, F)$ is positive for $\lambda_s \leq \lambda_{sk}^{(3)}$ and it is negative for $\lambda_s \in \left(\lambda_{sk}^{(3)}, 3t_s/\beta\right)$, where $\lambda_{sk}^{(3)} = \min\left(3t_s/\beta, \lambda_{sk}^{(2)}\right)$. Solving for λ_s such that $\Delta(F, F) = 0$, we obtain $\lambda_{sk}^{(2)}$ in (20). By (20), $\lambda_{sk}^{(2)}$ is increasing in λ_f, and $\lim_{\lambda_f \downarrow 0} \lambda_{sk}^{(2)} = \frac{3t_s}{\beta}\left(\sqrt{1 + \frac{\beta t_f}{t_s}} - 1\right)$ and $\lim_{\lambda_f \uparrow \infty} \lambda_{sk}^{(2)} = \infty$.

Second, consider, $\lambda_s \geq 3t_s/\beta$. By (14) and (18),

$$\Delta(F, F) = \frac{t_s}{2} + \frac{\beta\lambda_f\lambda_s}{2t_f} + \frac{\beta t_f}{2} + \frac{\beta^2\lambda_f\lambda_s}{2t_s} - (v_s - t_s - c).$$

In this case, the brand that does not license (e.g., assume brand B without loss of generality) is a monopoly in market s when only one brand licenses and $\Delta(F, F) < 0$ since we assume that a brand prefers being a monopoly in market s

when the other brand licenses over competing the other brand in market s and market f when they both license (i.e., v_s is significantly large). □

Proof of Proposition 2: Using Lemma 2, we characterize the equilibrium strategies of brands with fixed-fee contract. Recall that we define $\lambda_{sk}^L = \min(\lambda_{sk}^{(1)}, \lambda_{sk}^{(3)})$ and $\lambda_{sk}^H = \max(\lambda_{sk}^{(1)}, \lambda_{sk}^{(3)})$. In addition, as in the proof of Lemma 2, we let $\Delta(F, NL)$ and $\Delta(F, F)$, respectively, be the benefit to a brand from licensing via fixed-fee contract when the other brand does not license and when the other brand licenses via fixed-fee contract. Recall by Lemma 2 that $\Delta(F, NL) > 0$ if, and only if, $\lambda_s < \lambda_{sk}^{(1)}$, and $\Delta(F, F) > 0$ if, and only if, $\lambda_s < \lambda_{sk}^{(3)}$, which we will use below. We prove each part of the proposition separately.

Part I: In this part, $v_f \le t_f + t_s/2\beta$so that $\lambda_{sk}^{(1)} < \infty$ by (19). Now we consider four cases: (a) $\lambda_s \le \lambda_{sk}^L$, (b) $\lambda_{sk}^{(1)} < \lambda_s < \lambda_{sk}^{(3)}$, (c) $\lambda_{sk}^{(3)} < \lambda_s < \lambda_{sk}^{(1)}$, and (d) $\lambda_s \ge \lambda_{sk}^H$.

Case I(a): In this case, $\Delta(F, NL) > 0$ and $\Delta(F, F) > 0$, that is, it is optimal for a brand to license independent from whether the other brand licenses or not. Thus, both brands license and use fixed-fee contract for $\lambda_s \le \lambda_{sk}^L$.

Case I(b): In this case, $\lambda_{sk}^{(1)} \le \lambda_{sk}^{(3)}$ so that $\lambda_{sk}^L = \lambda_{sk}^{(1)}$ and $\lambda_{sk}^H = \lambda_{sk}^{(3)}$. Then, for $\lambda_s \in (\lambda_{sk}^L, \lambda_{sk}^H)$, $\Delta(F, NL) \le 0$ and $\Delta(F, F) > 0$; therefore, it is optimal for a brand to license when the other brand uses fixed-fee contract and to not license when the other brand does not license. In other words, the best response of a brand is to use the same strategy as the other brand. Thus, there are two Nash equilibria in this case: (i) both brands use fixed-fee contract, and (ii) no brand licenses.

Case I(c): In this case, $\lambda_{sk}^{(1)} > \lambda_{sk}^{(3)}$ so that $\lambda_{sk}^L = \lambda_{sk}^{(3)}$ and $\lambda_{sk}^H = \lambda_{sk}^{(1)}$. Thus, for $\lambda_s \in (\lambda_{sk}^L, \lambda_{sk}^H)$, $\Delta(F, NL) > 0$ and $\Delta(F, F) \le 0$; therefore, it is optimal for a brand to not license when the other brand uses fixed-fee contract and to use fixed-fee contract when the other brand does not license. In other words, the best response of each brand is to use the opposite strategy of the other brand. As a result, only one brand uses fixed-fee contract in this case.

Case I(d): In this case, $\Delta(F, NL) \le 0$ and $\Delta(F, F) \le 0$, which implies that it is optimal for a brand not to license independent from whether the other brand licenses or not. Thus, both brands do not license.

Part II: Lastly, we assume $v_f > t_f + t_s/2\beta$ and prove the second part of the proposition. In this case, $\lambda_{sk}^{(1)} = \infty$ by (19). In this case, $\lambda_{sk}^L = \lambda_{sk}^{(3)}$ and $\lambda_{sk}^H = \lambda_{sk}^{(1)} = \infty$. For $\lambda_s \le \lambda_{sk}^L$, $\Delta(F, NL) > 0$ and $\Delta(F, F) > 0$ by Lemma 2. Hence each brand always prefers using fixed-fee contract whether the other brand licenses or not, that is, both brands license. However, for $\lambda_s > \lambda_{sk}^L$, $\Delta(F, NL) > 0$ and $\Delta(F, F) \le 0$ by Lemma 2, that is, the best response of a brand is to use the fixed-fee contract when the other brand does not license and to not license when the other brand licenses. Therefore, for $\lambda_s > \lambda_{sk}^L$, either brand A or brand B licenses and only one brand uses fixed-fee contract. □

Proof of Corollary 2: By (19), $\lambda_{sk}^{(1)}$ is increasing in λ_f and $\lim_{\lambda_f \uparrow \infty} \lambda_{sk}^{(1)} = 3t_s/\beta$. Hence, $\lambda_{sk}^{(1)} < 3t_s/\beta$ for $v_f \le t_f + t_s/2\beta$. In addition by (20), $\lambda_{sk}^{(2)}$ is increasing in λ_f, and $\lim_{\lambda_f \downarrow 0} \lambda_{sk}^{(2)} = \frac{3t_s}{\beta}\left(\sqrt{1 + \frac{\beta t_f}{t_s}} - 1\right)$. Therefore, if $\lim_{\lambda_f \downarrow 0} \lambda_{sk}^{(2)} > 3t_s/\beta$, $\lambda_{sk}^{(3)} = \min\left(3t_s/\beta, \lambda_{sk}^{(2)}\right) > \lambda_{sk}^{(1)}$ for all λ_f, and by Proposition 2(Ic), it is never the case that only one brand licenses in equilibrium. □

Proof of Lemma 3: Since both brands license in this case as in Section 5.2, marginal snob θ_s and follower θ_f are also given by (10) and (11), respectively. In addition, $D_s^A = 1 - D_s^B = \theta_s$ and $D_f^a = D_f^b = \theta_f$ by rational expectations. Using this and substituting $T^A = r^A \beta D_f^a$, and $T^B = r^B \beta D_f^b$ into (3)-(4), we obtain profits of brands and their licensees as follows:

$$\Pi^A\left(p^A, r^A\right) = \left(p^A - c\right) \frac{-t_f + p^B t_f + t_f t_s + \beta\lambda_f\lambda_s + \beta\lambda_s p^a - \beta\lambda_s p^b}{2t_f t_s + 2\beta\lambda_f\lambda_s}$$

$$
\begin{aligned}
&+r^A\beta\frac{-\lambda_f p^A+\lambda_f p^B-p^a t_s+p^b t_s+t_f t_s+\beta\lambda_f\lambda_s}{2t_f t_s+2\beta\lambda_f\lambda_s}\\
\Pi^B\left(p^B,r^B\right) &= \left(p^B-c\right)\left(1-\frac{-t_f+p^B t_f+t_f t_s+\beta\lambda_f\lambda_s+\beta\lambda_s p^a-\beta\lambda_s p^b}{2t_f t_s+2\beta\lambda_f\lambda_s}\right)\\
&+r^B\beta\left(1-\frac{-\lambda_f p^A+\lambda_f p^B-p^a t_s+p^b t_s+t_f t_s+\beta\lambda_f\lambda_s}{2t_f t_s+2\beta\lambda_f\lambda_s}\right)\\
\Pi^a\left(p^a\right) &= \left(p^a-r^A\right)\beta\frac{-\lambda_f p^A+\lambda_f p^B-p^a t_s+p^b t_s+t_f t_s+\beta\lambda_f\lambda_s}{2t_f t_s+2\beta\lambda_f\lambda_s}\\
\Pi^b\left(p^b\right) &= \left(p^b-r^B\right)\beta\left(1-\frac{-\lambda_f p^A+\lambda_f p^B-p^a t_s+p^b t_s+t_f t_s+\beta\lambda_f\lambda_s}{2t_f t_s+2\beta\lambda_f\lambda_s}\right).
\end{aligned}
$$

Using first-order conditions, we obtain prices for given royalty fees as follows:

$$
\begin{aligned}
p^A &= c+t_s+\frac{\beta\lambda_f}{t_f}\left(\lambda_s-r^A\right)+\beta\left(r^A-r^B\right)\frac{2\beta\lambda_f^2\lambda_s+3\lambda_f t_f t_s+\lambda_s t_f t_s}{9t_f^2 t_s+4\beta\lambda_f\lambda_s t_f}\\
p^B &= c+t_s+\frac{\beta\lambda_f}{t_f}\left(\lambda_s-r^B\right)+\beta\left(r^B-r^A\right)\frac{2\beta\lambda_f^2\lambda_s+3\lambda_f t_f t_s+\lambda_s t_f t_s}{9t_f^2 t_s+4\beta\lambda_f\lambda_s t_f}\\
p^a &= t_f+r^A+\frac{\beta\lambda_f\lambda_s}{t_s}+\left(r^B-r^A\right)\frac{3t_f t_s+2\beta\lambda_f\lambda_s-\beta\lambda_f^2}{9t_f t_s+4\beta\lambda_f\lambda_s}\\
p^b &= t_f+r^B+\frac{\beta\lambda_f\lambda_s}{t_s}+\left(r^A-r^B\right)\frac{3t_f t_s+2\beta\lambda_f\lambda_s-\beta\lambda_f^2}{9t_f t_s+4\beta\lambda_f\lambda_s}.
\end{aligned}
$$

Substituting above prices in brand A's profit function, we obtain:

$$
\begin{aligned}
\Pi^A\left(r^A\right) = {} & \beta\left(r^A-r^B\right)\frac{2\beta\lambda_f^2\lambda_s+3\lambda_f t_f t_s+\lambda_s t_f t_s}{9t_f^2 t_s+4\beta\lambda_f\lambda_s t_f}\left(\frac{1}{2}+\beta\left(r^A-r^B\right)\frac{2\beta\lambda_f^2\lambda_s+3\lambda_f t_f t_s+\lambda_s t_f t_s}{2\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)}\right)\\
& \left(t_s+\frac{\beta\lambda_f}{t_f}\left(\lambda_s-r^A\right)\right)\left(\frac{1}{2}+\beta\left(r^A-r^B\right)\frac{2\beta\lambda_f^2\lambda_s+3\lambda_f t_f t_s+\lambda_s t_f t_s}{2\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)}\right)\\
& +r^A\beta\left(\frac{1}{2}+t_s\left(r^A-r^B\right)\frac{\beta\left(\lambda_f-\lambda_s\right)^2-3t_f t_s-\beta\lambda_s^2}{2\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)}\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\frac{d\Pi^A\left(r^A\right)}{dr^A} = {} & \frac{\beta}{2}\left(\frac{-3\lambda_f t_s+2\lambda_s t_s+9t_f t_s+4\beta\lambda_f\lambda_s}{9t_f t_s+4\beta\lambda_f\lambda_s}+r^B t_s\frac{2\beta\lambda_s^2\left(2\beta\lambda_f^2-t_f t_s\right)+\beta\lambda_f\lambda_s\left(27t_f t_s+2\beta\lambda_f^2\right)+27t_f^2 t_s^2}{\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)^2}\right.\\
& \left.-2r^A\frac{4\beta^3\lambda_f^4\lambda_s^2+14\beta^2\lambda_f^3\lambda_s t_f t_s+8\beta^2\lambda_f^2\lambda_s^2 t_f t_s+9\beta\lambda_f^2 t_f^2 t_s^2+33\beta\lambda_f\lambda_s t_f^2 t_s^2-\beta\lambda_s^2 t_f^2 t_s^2+27t_f^3 t_s^3}{t_f\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)^2}\right).
\end{aligned}
$$

Note that

$$
\frac{d^2\Pi^A\left(r^A\right)}{d{r^A}^2} = -\beta\frac{\left(4\beta^2\lambda_f^4+8\beta\lambda_f^2 t_f t_s-t_f^2 t_s^2\right)\lambda_s^2+\beta\lambda_f\lambda_s t_f t_s\left(33t_f t_s+14\beta\lambda_f^2\right)+9\beta\lambda_f^2 t_f^2 t_s^2+27t_f^3 t_s^3}{t_f\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)^2}.
$$

Observe that $d^2\Pi^A\left(r^A\right)/d{r^A}^2<0$ and $\lim_{r^A\uparrow\infty} d\Pi^A\left(r^A\right)/dr^A<0$ by our assumption that $\lambda_f\geq\sqrt{t_f t_s/2\beta}$. Also we have

$$
\begin{aligned}
\lim_{r^A\downarrow 0}\frac{d\Pi^A\left(r^A\right)}{dr^A} = {} & \frac{\beta}{2}\left(\frac{2\lambda_s\left(2\beta\lambda_f+t_s\right)+3t_s\left(3t_f-\lambda_f\right)}{9t_f t_s+4\beta\lambda_f\lambda_s}\right.\\
& \left.+r^B t_s\frac{2\beta\lambda_s^2\left(2\beta\lambda_f^2-t_f t_s\right)+\beta\lambda_f\lambda_s\left(27t_f t_s+2\beta\lambda_f^2\right)+27t_f^2 t_s^2}{\left(t_f t_s+\beta\lambda_f\lambda_s\right)\left(9t_f t_s+4\beta\lambda_f\lambda_s\right)^2}\right).
\end{aligned}
\tag{54}
$$

Note by $\lambda_f\geq\sqrt{t_f t_s/2\beta}$, the second term inside the parenthesis above is always positive. Consider two cases: (i) $\lambda_f\leq 3t_f$, or $\lambda_s\geq\lambda_{sr}^{(1)}$ when $\lambda_f>3t_f$, and (ii) $\lambda_s<\lambda_{sr}^{(1)}$ when $\lambda_f>3t_f$, where $\lambda_{sr}^{(1)}$ is given by (41).

Case (i): In this case, $\lim_{r_A \downarrow 0} d\Pi^A(r^A)/dr^A > 0$ by (54). This by $\Pi^A(r^A)$ being concave and $\lim_{r_A \uparrow \infty} d\Pi^A(r^A)/dr^A < 0$ implies that there exists unique $r^A \in (0,\infty)$ that satisfies first-order condition, i.e., $d\Pi^A(r^A) = 0$. Similarly, by symmetry, there exists a unique $r^B \in (0,\infty)$ that satisfies $d\Pi^B(r^B) = 0$. Solving $d\Pi^I(r^I) = 0$ for r^I $(I = A, B)$, the optimal royalty fee for brand I $(I = A, B)$ is unique and given by:

$$r^I = t_f \frac{2\lambda_s(2\beta\lambda_f + t_s) + 3t_s(3t_f - \lambda_f)}{3t_f t_s + 2\beta\lambda_f^2}.$$

Case (i): In this case, by (54), $\lim_{r_A \downarrow 0} d\Pi^A(r^A)/dr^A \leq 0$ if, and only if, $r^B \leq \bar{r}$, where $\bar{r}$ is positive in this case and given by:

$$\bar{r} = \frac{(3t_s(\lambda_f - 3t_f) - 2\lambda_s(t_s + 2\beta\lambda_f))(t_f t_s + \beta\lambda_f\lambda_s)(9t_f t_s + 4\beta\lambda_f\lambda_s)}{t_s\left(27t_f^2 t_s^2 + 2\beta^2\lambda_f^3\lambda_s + 4\beta^2\lambda_f^2\lambda_s^2 - 2\beta\lambda_s^2 t_f t_s + 27\beta\lambda_f\lambda_s t_f t_s\right)}.$$

Similarly, by symmetry $\lim_{r_B \downarrow 0} d\Pi^B(r^B)/dr^B \leq 0$ if, and only if, $r^B \leq \bar{r}$.

First let us characterize all equilibria where a brand sets its royalty fee less than or equal to $\bar{r}$. Without loss of generality, assume that brand B sets its royalty fee $r^B \leq \bar{r}$. The best response of brand A in this case is to set $r^A = 0$ since its profit is always decreasing in r^A (by $\lim_{r_A \downarrow 0} d\Pi^A(r^A)/dr^A \leq 0$ and $\Pi^A(r^A)$ being concave). When $r^A = 0$, best response of brand B is also to set $r^B = 0$ since $\bar{r} > r^A = 0$ and brand B's profit is always decreasing in r^B. This indicates that $r^I = 0$ $(I = A, B)$ is the only equilibrium where a brand sets $r^I < \bar{r}$ in this case.

Next we characterize all equilibria where a brand sets its royalty fee greater than $\bar{r}$. Again assume that brand B sets its royalty fee $r^B > \bar{r}$. Since $\lim_{r_A \downarrow 0} d\Pi^A(r^A)/dr^A > 0$ for $r^B > \bar{r}$, the best response of brand A in this case is to set its royalty fee $r^A > 0$ such that $d\Pi^A(r^A)/dr^A = 0$. First suppose that r^A satisfying $d\Pi^A(r^A)/dr^A = 0$ is less than or equal to $\bar{r}$, i.e., $r^A \leq \bar{r}$. By the discussion in above paragrapg, the best response of brand B is to set its royalty fee equal to zero, i.e., $r^B = 0$, when $r^A \leq \bar{r}$. This is a contradiction to our initial assumption that $r^B > \bar{r}$. Now suppose that r^A satisfying $d\Pi^A(r^A)/dr^A = 0$ is greater than $\bar{r}$, i.e., $r^A > \bar{r}$. In this case, since $\lim_{r_B \downarrow 0} d\Pi^B(r^B)/dr^B > 0$, $\lim_{r_B \uparrow \infty} d\Pi^B(r^B)/dr^B < 0$, and $Pi^B(r^B)$ is concave, brand B will set its royalty fee r^B such that $d\Pi^B(r^B)/dr^B = 0$. This implies that r^A and r^B must simultaneously satisfy $d\Pi^A(r^A)/dr^A = 0$ and $d\Pi^A(r^B)/dr^B = 0$. Solving for r^A and r^B, we obtain

$$r^A = r^B = t_f \frac{2\lambda_s(2\beta\lambda_f + t_s) + 3t_s(3t_f - \lambda_f)}{3t_f t_s + 2\beta\lambda_f^2} < 0$$

where the inequality follows from $\lambda_s < \lambda_{sr}^{(1)}$ and $\lambda_f > 3t_f$ in this case. However, note that this a contradiction to our assumption that brand B sets its royalty fee $r^B > \bar{r}$. Therefore, in case (ii), $r^I > \bar{r} > 0$ $(I = A, B)$ cannot be an equilibrium.

Summarizing above analysis, for $\lambda_f \geq 3t_f$ and $\lambda_s < \lambda_{sr}^{(1)}$, $r^I = 0$; otherwise,

$$r^I = t_f \frac{2\lambda_s(2\beta\lambda_f + t_s) + 3t_s(3t_f - \lambda_f)}{3t_f t_s + 2\beta\lambda_f^2}$$

for $I = A, B$. Then plugging $r^A = r^B$ above in price expressions, prices of brand I $(I = A, B)$ and its licensee i $(i = a, b)$ are, respectively, equal to $p^I = c + t_s + \beta\lambda_f(\lambda_s - r^I)/t_f$ and $p^i = r^I + t_f + \beta\lambda_f\lambda_s/t_s$. □

Proof of Lemma 4: We will prove each part of the lemma separately.

(i) In this part, we will show that both brands are better off when they both do not license compared to the case when they both license by using royalty contract if, and only if, $\lambda_s > \lambda_{sr}^{(2)}$ and for $\lambda_f \in (t_f, 3t_f)$. By Lemma 3, the royalty fee of brand I $(I = A, B)$ when both brands license by using royalty contract is given by:

$$r^I = \begin{cases} 0, & \text{if } \lambda_f \geq 3t_f \text{ and } \lambda_s < \lambda_{sr}^{(1)}, \\ t_f \frac{2\lambda_s(2\beta\lambda_f + t_s) + 3t_s(3t_f - \lambda_f)}{3t_f t_s + 2\beta\lambda_f^2}, & \text{if otherwise.} \end{cases} \tag{55}$$

By (9), (22) and (55), $\Pi^I(R,R) > \Pi^I(NL,NL)$ for $I = A,B$ when $\lambda_f \geq 3t_f$ and $\lambda_s < \lambda_{sr}^{(1)}$, or $\lambda_f \leq t_f$.

Now consider all other cases, i.e., $\lambda_f \geq 3t_f$ and $\lambda_s \geq \lambda_{sr}^{(1)}$, or $\lambda_f \in (t_f, 3t_f)$. In all these cases, by (9), (22) and (55),

$$\Pi^I(R,R) - \Pi^I(NL,NL) = \frac{\beta}{2t_f}\left(\lambda_f\lambda_s - (\lambda_f - t_f)\, t_f \frac{2\lambda_s(2\beta\lambda_f + t_s) + 3t_s(3t_f - \lambda_f)}{3t_f t_s + 2\beta\lambda_f^2}\right)$$

for $I = A,B$. After some simplifications, we have

$$\Pi^I(R,R) - \Pi^I(NL,NL) = \frac{\beta}{2t_f\left(2\beta\lambda_f^2 + 3t_f t_s\right)}\left(g_1(\lambda_f)\lambda_s - h_1(\lambda_f)\right), \tag{56}$$

for $I = A,B$, where $g_1(\lambda_f)$ and $h_1(\lambda_f)$ are given, respectively, by (37) and (39). Note that $h_1(\lambda_f) > 0$ for $\lambda_f \in (t_f, 3t_f)$ and $h_1(\lambda_f) \leq 0$ for $\lambda_f \geq 3t_f$. Also note that $g_1(\lambda_f)$ is (strictly) convex for $\lambda_f > t_f$ and $\lim_{\lambda_f \downarrow t_f} dg_1(\lambda_f)/d\lambda_f > 0$ which indicates that $g_1(\lambda_f)$ is increasing for $\lambda_f > t_f$. This by $\lim_{\lambda_f \downarrow t_f} g_1(\lambda_f) > 0$ futher implies that $g_1(\lambda_f) > 0$ for $\lambda_f > t_f$.

Now consider $\lambda_f \in (t_f, 3t_f)$. In this case, $g_1(\lambda_f) > 0$ and $h_1(\lambda_f) > 0$, and by (56), both firms are better off from not licensing, i.e., $\Pi^I(NL,NL) > \Pi^I(R,R)$ for $I = A,B$, if $\lambda_s < \lambda_{sr}^{(2)}$, where $\lambda_{sr}^{(2)} = h_1(\lambda_f)/g_1(\lambda_f)$. Finally consider $\lambda_f \geq 3t_f$. In this case, $g_1(\lambda_f) > 0$ and $h_1(\lambda_f) \leq 0$, therefore, $\Pi^I(R,R) > \Pi^I(NL,NL)$ for $I = A,B$ by (56). □

(ii) In this part, we characterize cases where each brand is better off when both use fixed-fee contracts relative to the case when both use royalty contracts. Note by (13) and (22) that for $\lambda_f \geq t_f$ each brand is better off when they both license for free relative to the case that both license by using royalty contract, i.e., $\Pi^I(FL,FL) \geq \Pi^I(R,R)$ for $I = A,B$. Since $\Pi^I(F,F) > \Pi^I(FL,FL)$ $(I = A,B)$ by (13) and (14), it follows that for $\lambda_f \geq t_f$, each brand is better off when they both use fixed-fee contract compared to the case when they both use royalty contract.

Next, we consider $\lambda_f < t_f$. We will show that there exists $0 < \lambda_{fr}^L < \lambda_{fr}^H < t_f$ such that fixed-fee contract dominates for $\lambda_f \geq \lambda_{fr}^H$ and is dominated by royalty contract for $\lambda_f \leq \lambda_{fr}^L$, and for $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$, whether fixed-fee contract dominates or not depends on the value of λ_s. Using (14), (22) and (55), and through some algebra, we have

$$\Pi^I(F,F) - \Pi^I(R,R) = \frac{\beta}{2t_s\left(2\beta\lambda_f^2 + 3t_f t_s\right)}\left(g_2(\lambda_f)\lambda_s - h_2(\lambda_f)\right), \tag{57}$$

for $I = A,B$, where $g_2(\lambda_f)$ and $h_2(\lambda_f)$ are given, respectively, by (38) and (40). Note that $g_2(\lambda_f)$ is (strictly) convex in λ_f, and $\lim_{\lambda_f \downarrow 0} g_2(\lambda_f) < 0$ and $\lim_{\lambda_f \uparrow t_f} g_2(\lambda_f) > 0$. This indicates that there exists a unique $\lambda_{fr}^{(1)} \in (0, t_f)$ such that $g_2\left(\lambda_{fr}^{(1)}\right) = 0$, $g_2(\lambda_f) < 0$ for all $\lambda_f < \lambda_{fr}^{(1)}$ and $g_2(\lambda_f) > 0$ for all $\lambda_{fr}^{(1)} < \lambda_f < t_f$. Similarly $h_2(\lambda_f)$ is either (strictly) convex or concave in λ_f, and $\lim_{\lambda_f \downarrow 0} h_2(\lambda_f) > 0$ and $\lim_{\lambda_f \uparrow t_f} h_2(\lambda_f) < 0$. Thus, there exists a unique $\lambda_{fr}^{(2)} \in (0, t_f)$ such that $h_2\left(\lambda_{fr}^{(2)}\right) = 0$, $h_2(\lambda_f) > 0$ for all $\lambda_f < \lambda_{fr}^{(2)}$ and $h_2(\lambda_f) < 0$ for all $\lambda_{fr}^{(2)} < \lambda_f < t_f$. Define $\lambda_{fr}^L = \min(\lambda_{fr}^{(1)}, \lambda_{fr}^{(2)})$ and $\lambda_{fr}^H = \max(\lambda_{fr}^{(1)}, \lambda_{fr}^{(2)})$, and consider three cases: (a) $\lambda_f \leq \lambda_{fr}^L$, (b) $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$, (c) $\lambda_f \in \left(\lambda_{fr}^H, t_f\right)$.

Case (a): In this case, $g_2(\lambda_f) \leq 0$ and $h_2(\lambda_f) \geq 0$, and by (57), the fixed-fee contract is dominated by royalty contract, i.e., $\Pi^I(F,F) \leq \Pi^I(R,R)$ for $I = A,B$.

Case (b): If $\lambda_{fr}^{(1)} \leq \lambda_{fr}^{(2)}$, $\lambda_{fr}^L = \lambda_{fr}^{(1)}$ and $\lambda_{fr}^H = \lambda_{fr}^{(2)}$. For $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$, $g_2(\lambda_f) > 0$ and $h_2(\lambda_f) > 0$, and by (57), the fixed-fee contract dominates the royalty contract if, and only if, $\lambda_s > \lambda_{sr}^{(3)}$, where $\lambda_{sr}^{(3)} = h_2(\lambda_f)/g_2(\lambda_f)$. Similarly, if $\lambda_{fr}^{(1)} > \lambda_{fr}^{(2)}$, $g_2(\lambda_f) < 0$ and $h_2(\lambda_f) < 0$ for $\lambda_f \in \left(\lambda_{fr}^L, \lambda_{fr}^H\right)$, and the fixed-fee contract dominates the royalty contract if, and only if, $\lambda_s < \lambda_{sr}^{(3)}$.

Case (c): In this case, $g_2(\lambda_f) \geq 0$ and $h_2(\lambda_f) < 0$, or $g_2(\lambda_f) > 0$ and $h_2(\lambda_f) \leq 0$, and $\Pi^I(F,F) > \Pi^I(R,R)$ for $I = A, B$ by (57). Combined with $\Pi^I(F,F) > \Pi^I(R,R)$ for $\lambda_f \geq t_f$ for $I = A, B$, this implies that fixed-fee contract dominates the royalty contract for $\lambda_f \geq \lambda_{fr}^H$. $\square$

Proof of Lemma 5: The first part of Lemma 5 follows from Lemma 2(i) by (23) and (24). Next, we will prove the second part of the lemma. To that end, we will analyze the benefit to a brand from licensing via royalty contract when the other brand uses royalty contract. Note that $\Pi^A(R,R) = \Pi^B(R,R)$ by (22) and $\Pi^A(NL,R) = \Pi^B(NL,R)$ by (24). Therefore, the benefit from licensing via royalty contract when the other brand uses royalty contract is the same for brand A and B, (i.e., $\Pi^A(R,R) - \Pi^A(NL,R) = \Pi^B(R,R) - \Pi^B(NL,R)$). We let $\Delta(R,R)$ denote that benefit. Next we will consider two cases: (1) $\lambda_s \geq 3t_s/\beta$, and (2) $\lambda_s < 3t_s/\beta$.

<u>*Case 1:*</u> In this case, by (23) and (24), the brand that does not license when only one brand uses royalty contract is a monopoly in market s and $\Delta(R,R) \leq 0$ since we assume that a brand prefers being a monopoly in market s when the other brand licenses over competing the other brand in market s and market f when they both license (i.e., v_s is significantly large).

<u>*Case 2:*</u> In this case, we will consider two more sub-cases: (a) $\lambda_f \leq t_f$, and (b) $\lambda_f \geq 3t_f$.

Sub-case (2a): In this case (i.e., $\lambda_s < 3t_s/\beta$ and $\lambda_f \leq t_f$), by (22), (24) and (55), we have

$$\Delta(R,R) = \frac{\beta\Omega(\lambda_s)}{18t_f t_s\left(3t_f t_s + 2\beta\lambda_f^2\right)},$$

where

$$\Omega(\lambda_s) = -\beta t_f\left(3t_f t_s + 2\beta\lambda_f^2\right)\lambda_s^2 + 27t_f t_s^2(\lambda_f - 3t_f)(\lambda_f - t_f) + 3\lambda_f t_s\left(3t_f t_s + 2\beta\lambda_f^2 + 4\beta(\lambda_f - 3t_f)(\lambda_f - t_f)\right)\lambda_s. \quad (58)$$

Note that $\Omega(\lambda_s)$ is a concave quadratic function of λ_s and its discriminant is nonnegative for $\lambda_f \leq t_f$. Thus $\Omega(\lambda_s) = 0$ has two real roots. Moreover, for $\lambda_f \leq t_f$, its smaller root is always negative and its bigger root $\lambda_{sr}^{(5)}$ is always positive and given by (44). Then it follows that $\Omega(\lambda_s) > 0$ if, and only if, $\lambda_s < \lambda_{sr}^{(4)}$, where $\lambda_{sr}^{(4)} = \min\left(3t_s/\beta, \lambda_{sr}^{(5)}\right)$ when $\lambda_f \leq t_f$.

Sub-case (2b): Note by (41) that $\lambda_{sr}^{(1)} \in (0, 3t_s/\beta)$ for $\lambda_f \geq 3t_f$ since it is increasing in λ_f and $\lim_{\lambda_f \uparrow \infty} \lambda_{sr}^{(1)} = 3t_s/4\beta$. First consider $\lambda_f \geq 3t_f$ and $\lambda_s < \lambda_{sr}^{(1)}$. In this case, when both brands license, the optimal royalty fee is equal to zero by (55) and hence by (22) and (24), we have $\Delta(R,R) = \frac{\beta\lambda_s}{18t_f t_s}\phi(\lambda_s)$, where

$$\phi(\lambda_s) = 9\lambda_f t_s - 6t_f t_s - \beta\lambda_s t_f.$$

Note that $\phi(\lambda_s)$ is decreasing in λ_s and $\lim_{\lambda_s \uparrow 3t_s/\beta} \phi(\lambda_s) > 0$ for $\lambda_f \geq 3t_f$. This implies that $\Delta(R,R) > 0$ for $\lambda_f \geq 3t_f$ and $\lambda_s < \lambda_{sr}^{(1)}$.

Now consider $\lambda_f \geq 3t_f$ and $\lambda_s \geq \lambda_{sr}^{(1)}$. In this case, by (22) and (24), we have

$$\Delta(R,R) = \frac{\beta\Omega(\lambda_s)}{18t_f t_s\left(3t_f t_s + 2\beta\lambda_f^2\right)},$$

where $\Omega(\lambda_s)$ is given by (58). For $\lambda_f \geq 3t_f$, the discriminant of quadratic function $\Omega(\lambda_s)$ is nonnegative and hence $\Omega(\lambda_s) = 0$ has two real roots. In addition, the smaller root is always negative and the bigger root $\lambda_{sr}^{(5)}$ is always positive in this case. Moreover, by (44), the bigger root $\lambda_{sr}^{(5)}$ is always greater than $3t_s/\beta$. Then it follows by $\Omega(\lambda_s)$

being a concave quadratic function that $\Delta(R,R) > 0$ if, and only if, $\lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$ and $\lambda_s \geq \lambda_{sr}^{(1)}$. Thus $\Delta(R,R) > 0$ in this case.

To summarize, $\Delta(R,R) \leq 0$: (i) if, and only if, $\lambda_s \geq \lambda_{sr}^{(4)} = \min\left(3t_s/\beta, \lambda_{sr}^{(5)}\right)$ when $\lambda_f \leq t_f$ by Case 1 and Case 2a; (ii) if $\lambda_s \geq 3t_s/\beta$ when $t_f < \lambda_f < 3t_f$ by Case 1; and (iii) if, and only if, $\lambda_s \geq 3t_s/\beta$ when $\lambda_f \geq 3t_f$ by Case 1 and Case 2b. Hence, the second part of the lemma follows. □

Proof of Proposition 3: As in the proof of Lemma 5, we define $\Delta(R,NL)$ and $\Delta(R,R)$, respectively, as the benefit to a brand from using royalty contract when the other brand does not license and when the other brand licenses via royalty contract. By Lemma 5(i), $\Delta(R,NL) > 0$ for $\lambda_s < \lambda_{sk}^{(1)}$ and $\Delta(R,NL) \leq 0$ for $\lambda_s \geq \lambda_{sk}^{(1)}$, where $\lambda_s \geq \lambda_{sk}^{(1)}$ is given by (19). By Lemma 5(ii), $\Delta(R,R) \leq 0$: (i) if, and only if, $\lambda_s \geq \lambda_{sr}^{(4)} = \min\left(3t_s/\beta, \lambda_{sr}^{(5)}\right)$ when $\lambda_f \leq t_f$; (ii) if $\lambda_s \geq 3t_s/\beta$ when $t_f < \lambda_f < 3t_f$; and (iii) if, and only if, $\lambda_s \geq 3t_s/\beta$ when $\lambda_f \geq 3t_f$, where $\lambda_{sr}^{(5)}$ is given by (44). We will use this below. Also let $\lambda_{sr}^{L} = \min(\lambda_{sk}^{(1)}, \lambda_{sr}^{(4)})$ and $\lambda_{sr}^{H} = \max(\lambda_{sk}^{(1)}, \lambda_{sr}^{(4)})$. Next, we prove each part of the proposition separately.

Part I: In this part, assuming $v_f \leq t_f + t_s/2\beta$, we prove the first part of the proposition. In this case, by (19), $\lambda_{sk}^{(1)} < 3t_s/\beta$ since it is increasing in λ_f and goes to $3t_s/\beta$ as λ_f goes to ∞. Next, consider four cases: (a) $\lambda_s \geq \lambda_{sr}^{H}$ when $\lambda_f \leq t_f$, or $\lambda_s \geq 3t_s/\beta$ when $\lambda_f > t_f$; (b) $\lambda_{sk}^{(1)} < \lambda_s < \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_{sk}^{(1)} < \lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$; (c) $\lambda_{sr}^{(4)} < \lambda_s < \lambda_{sk}^{(1)}$ when $\lambda_f \leq t_f$; and (d) $\lambda_s < \lambda_{sr}^{L}$ when $\lambda_f \leq t_f$, or $\lambda_s < 3t_s/\beta$ when $\lambda_f \geq 3t_f$.

Case I(a): In this case, $\Delta(R,NL) \leq 0$ by Lemma 5(i) and $\Delta(R,R) \leq 0$ by Lemma 5(ii). Thus, a brand does not license no matter what the other brand does, i.e., both brands do not license.

Case I(b): By Lemma 5, $\Delta(R,NL) \leq 0$ and $\Delta(R,R) > 0$ in this case. Thus, the best response of a brand is to use the same licensing strategy as the other brand and there are two Nash equilibria: (i) both brands use royalty contract, and (ii) no brand licenses.

Case I(c): By Lemma 5, $\Delta(R,NL) > 0$ and $\Delta(R,R) \leq 0$ in this case. Then it follows that the best response of a brand is to not license when the other brand licenses and to license when the other brand does not license. That is, the best response is to play the opposite strategy of the other brand. As a result, only one brand licenses by using royalty contract while the other does not license.

Case I(d): In this case, using Lemma 5, $\Delta(R,NL) > 0$ and $\Delta(R,R) > 0$, and it is always beneficial for a brand to license by using royalty contract no matter what the other brand does. Thus, both brands license by using royalty contract.

Part II: In this part, $v_f > t_f + t_s/2\beta$ so that $\lambda_{sk}^{(1)} = \infty$ by (19). Thus $\Delta(R,NL) > 0$ by Lemma 5(i). For $\lambda_s \geq \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_s \geq 3t_s/\beta$ when $\lambda_f > t_f$, we have $\Delta(R,R) < 0$ by Lemma 5(ii); therefore, only one brand licenses. For $\lambda_s < \lambda_{sr}^{(4)}$ when $\lambda_f \leq t_f$, or $\lambda_s < 3t_s/\beta$ when $\lambda_f > 3t_f$, we have $\Delta(R,R) > 0$ from Lemma 5(ii) and thus both brands license. □

Proof of Lemma 6: The first part of the lemma follows from comparing $\Pi^I(NL,NL)$ in (9) and $\Pi^I(U,U)$ in (28). Next, we prove the second part of the lemma. By (14) and (28), we have

$$\Pi^A(U,U) - \Pi^I(F,F) = \frac{\beta}{2}\left(\lambda_s - \lambda_f\left(1 + \lambda_s\left(\frac{1}{t_f} + \frac{\beta}{t_s}\right)\right)\right)$$

for $I = A,B$. Then it follows that $\Pi^A(U,U) > \Pi^I(F,F)$ if, and only if, $\lambda_f < \lambda_{fc}^{(1)}$, where $\lambda_{fc}^{(1)}$ is given by (47). □

Proof of Lemma 7: We will prove each part of the lemma separately.

(i) In this part, we will characterize cases where it is beneficial for a brand to extend in the case when the other brand does not extend. To that end, we define $\Delta(U,NL)$ as the benefit to a brand from umbrella branding when the other brand does not extend in market f. By symmetry, $\Delta(U,NL)$ is the same for brand A and B. Now we consider three cases: (1) $\lambda_s < (\lambda_f - 3t_s/\beta)^+$, (2) $(\lambda_f - 3t_s/\beta)^+ < \lambda_s < \lambda_f + 3t_s/\beta$, and (3) $\lambda_s \geq \lambda_f + 3t_s/\beta$, where $(x)^+$.

Case (i.1): Note that this case occurs only for $\lambda_f > 3t_s/\beta$. In this case, if a brand decides to extend given that the other brand does not, it will be a monopoly in both markets. Since we assume a brand prefers being monopoly in market s to competing the other brand in market s (since we assume v_s is sufficiently large), it will always be beneficial for a brand to extend when the other brand does not, i.e., $\Delta(U,NL) > 0$.

Case (i.2): In this case, by (9) and (29), we have

$$\Delta(U,NL) = \frac{t_s}{2}\left(1 + \frac{\beta(\lambda_f - \lambda_s)}{3t_s}\right)^2 + \beta(v_f - t_f) - \frac{t_s}{2}$$

Note that $\lim_{\lambda_s \downarrow (\lambda_f - 3t_s/\beta)^+} \Delta(U,NL) > 0$, $\lim_{\lambda_s \uparrow \lambda_f + 3t_s/\beta} \Delta(U,NL) = (v_f - t_f)\beta - t_s/2$, and $d\Delta(U,NL)/d\lambda_s < 0$. This implies that $\Delta(U,NL) > 0$ when $v_f - t_f \geq t_s/2\beta$; and there exists $\check{\lambda}_{sc}^{(1)} \in \left((\lambda_f - 3t_s/\beta)^+, \lambda_f + 3t_s/\beta\right)$ such that $\Delta(U,NL) = 0$ at $\lambda_s = \check{\lambda}_{sc}^{(1)}$ and $\Delta(U,NL) > 0$ if, and only if, $\lambda_s < \check{\lambda}_{sc}^{(1)}$ when $v_f - t_f < t_s/2\beta$. Solving $\Delta(U,NL) = 0$ for λ_s, we obtain

$$\check{\lambda}_{sc}^{(1)} = \lambda_f + \frac{3t_s}{\beta} - \frac{3t_s}{\beta}\sqrt{1 - 2\beta\frac{v_f - t_f}{t_s}}$$

Case (i.3): In this case, if a brand decides to extend when the other brand does not, the brand will be a monopoly in market f and while the other brand will be a monopoly in market s. Thus, we have $\Delta(U,NL) = (v_f - t_f)\beta - t_s/2$ and $\Delta(U,NL) > 0$ if $v_f - t_f \geq t_s/2\beta$ and $\Delta(U,NL) < 0$ if $v_f - t_f < t_s/2\beta$.

In summary, $\Delta(U,NL) > 0$ for all λ_f and λ_s when $v_f - t_f \geq t_s/2\beta$; and $\Delta(U,NL) > 0$ if, and only if, $\lambda_s < \check{\lambda}_{sc}^{(1)}$ when $v_f - t_f < t_s/2\beta$. That is, $\Delta(U,NL) > 0$ if, and only if, $\lambda_s < \lambda_{sc}^{(1)}$, where $\lambda_{sc}^{(1)}$ is given by (48).

(ii) In this part, we will identify cases where it is beneficial for a brand to extend when the other brand extends. Then using those cases, we will prove the second part of the lemma. We define $\Delta(U,U)$ as the benefit to a brand from extending when the other brand extends. Note that for $\lambda_s \geq \lambda_f + 3t_s/\beta$ that the brand that does not extend when the other brand extends is a monopoly in market s. Since we assume brands prefer being a monopoly in market s to competing the other brand in both markets (i.e., v_s is sufficiently large), $\Delta(U,U) < 0$ for $\lambda_s \geq \lambda_f + 3t_s/\beta$. Next we analyze $\lambda_s < \lambda_f + 3t_s/\beta$ considering two cases: (1) $\lambda_f < 3t_s/\beta$, and (2) $\lambda_f \geq 3t_s/\beta$.

Case ii.1: In this case, by (28) and (30), we have

$$\Delta(U,U) = \frac{t_s}{2} + \beta\frac{\lambda_s - \lambda_f + t_f}{2} - \frac{t_s}{2}\left(1 - \frac{\beta(\lambda_f - \lambda_s)}{3t_s}\right)^2.$$

Note that $\lim_{\lambda_s \uparrow \lambda_f + 3t_s/\beta} \Delta(U,U) > 0$ and

$$\lim_{\lambda_s \downarrow 0} \Delta(U,U) = \frac{\beta}{18t_s}\left(9t_f t_s - \beta\lambda_f^2 - 3t_s\lambda_f\right).$$

Morever, $\Delta(U,U)$ is concave in λ_s, and it is first increasing and then decreasing in $\lambda_s \in [0, \lambda_f + 3t_s/\beta]$. Notice that $\lim_{\lambda_s \downarrow 0}\Delta(U,U) \geq 0$ when $t_f \geq 2t_s/\beta$; and when $t_f < 2t_s/\beta$, $\lim_{\lambda_s \downarrow 0}\Delta(U,U) > 0$ if, and only if, $\lambda_f < \check{\lambda}_{fc}^{(1)} \in (0, 3t_s/\beta)$, where

$$\check{\lambda}_{fc}^{(1)} = \frac{3t_s}{2\beta}\left(\sqrt{1 + 4\beta\frac{t_f}{t_s}} - 1\right).$$

When $t_f < 2t_s/\beta$ and $\lambda_f > \check{\lambda}_{fc}^{(1)}$, there exists $\check{\lambda}_{sc}^{(2)} \in (0, \lambda_f + 3t_s/\beta)$ such that $\Delta(U,U) = 0$ at $\lambda_s = \check{\lambda}_{sc}^{(2)}$ and $\Delta(U,U) > 0$ if, and only if, $\lambda_s > \check{\lambda}_{sc}^{(2)}$. Solving $\Delta(U,U) = 0$ for λ_s, we obtain

$$\check{\lambda}_{sc}^{(2)} = \lambda_f + \frac{3t_s}{2\beta} - \frac{3t_s}{2\beta}\sqrt{1 + 4\beta\frac{t_f}{t_s}}.$$

This implies that in this case (i.e., $\lambda_f < 3t_s/\beta$ and $\lambda_s < \lambda_f + 3t_s/\beta$), $\Delta(U,U) \leq 0$ if, and only if, $\lambda_s \leq \check{\lambda}_{sc}^{(2)}$, $t_f < 2t_s/\beta$ and $\lambda_f \in \left(\check{\lambda}_{fc}^{(1)}, 3t_s/\beta\right)$. This combined with $\Delta(U,U) < 0$ for $\lambda_s \geq \lambda_f + 3t_s/\beta$ implies that $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\max\left(\check{\lambda}_{sc}^{(2)}, 0\right), \lambda_f + 3t_s/\beta\right)$ when $t_f < \frac{t_s}{2\beta}$, or $\lambda_s \in (0, \lambda_f + 3t_s/\beta)$ when $t_f \geq \frac{t_s}{2\beta}$. By (49), $\lambda_{sc}^{(2)} = \max\left(\check{\lambda}_{sc}^{(2)}, 0\right)$ for $t_f < t_s/2\beta$, and $\lambda_{sc}^{(2)} = (\lambda_f - t_f - t_s/\beta)^+ = 0$ for $t_f \geq t_s/2\beta$ in this case (since $\lambda_f < 3t_s/\beta$). Thus, $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\lambda_{sc}^{(2)}, \lambda_{sc}^{(3)}\right)$ when $\lambda_f < 3t_s/\beta$, where $\lambda_{sc}^{(2)}$ and $\lambda_{sc}^{(3)}$ are given, respectively, by (49) and (50).

Case ii.2: In this case, assuming $\lambda_f \geq 3t_s/\beta$, we consider two sub-cases: (a) $\lambda_s < \lambda_f - 3t_s/\beta$, and (b) $\lambda_f - 3t_s/\beta < \lambda_s < \lambda_f + 3t_s/\beta$.

Subcase (ii.2a): In this subcase, when only one brand extends, the one that does not extend has zero profits as the other brand is a monopoly in both markets. If a brand decides to extend when the other brand already extends, by (28), the net benefit to that brand will be given by:

$$\Delta(U,U) = \frac{t_s}{2} + \beta\frac{\lambda_s - \lambda_f + t_f}{2}.$$

Note that $\Delta(U,U) > 0$ if, and only if, $\lambda_s > \lambda_f - t_f - t_s/\beta$. This implies that $\Delta(U,U) < 0$ when $t_f < 2t_s/\beta$, and $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in ((\lambda_f - t_f - t_s/\beta)^+, \lambda_f - 3t_s/\beta)$ when $t_f \geq 2t_s/\beta$.

Subcase (ii.2b): In this subcase, by (28) and (30), we have

$$\Delta(U,U) = \frac{t_s}{2} + \beta\frac{\lambda_s - \lambda_f + t_f}{2} - \frac{t_s}{2}\left(1 - \frac{\beta(\lambda_f - \lambda_s)}{3t_s}\right)^2.$$

Note that $\lim_{\lambda_s \downarrow \lambda_f - 3t_s/\beta} \Delta(U,U) = \beta t_f/2 - t_s$, $\lim_{\lambda_s \uparrow \lambda_f + 3t_s/\beta} \Delta(U,U) > 0$, and $d\Delta(U,U)/d\lambda_s > 0$. This implies that $\Delta(U,U) > 0$ when $t_f \geq 2t_s/\beta$, and when $t_f < 2t_s/\beta$, $\Delta(U,U) > 0$ if, and only if, $\lambda_s > \check{\lambda}_{sc}^{(2)}$.

In summary, from subcases (ii.2a) and (ii.2b), we have $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\max(\check{\lambda}_{sc}^{(2)}, 0), \lambda_f + 3t_s/\beta\right)$ when $t_f < 2t_s/\beta$. This combined with $\Delta(U,U) < 0$ for $\lambda_s \geq \lambda_f + 3t_s/\beta$ implies that when $t_f < 2t_s/\beta$ and $\lambda_f \geq 3t_s/\beta$, $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\lambda_{sc}^{(2)}, \lambda_{sc}^{(3)}\right)$, where $\lambda_{sc}^{(2)} = \max(\check{\lambda}_{sc}^{(2)}, 0)$ for $t_f < 2t_s/\beta$ by (49) and $\lambda_{sc}^{(3)} = \lambda_f + 3t_s/\beta$ by (50). In addition, from subcases (ii.2a) and (ii.2b), we have that $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in ((\lambda_f - t_f - t_s/\beta)^+, \lambda_f + 3t_s/\beta)$ when $t_f \geq 2t_s/\beta$. Together with $\Delta(U,U) < 0$ for $\lambda_s \geq \lambda_f + 3t_s/\beta$, it follows that when $t_f \geq 2t_s/\beta$ and $\lambda_f \geq 3t_s/\beta$, $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\lambda_{sc}^{(2)}, \lambda_{sc}^{(3)}\right)$, where $\lambda_{sc}^{(2)} = (\lambda_f - t_f - t_s/\beta)^+$ for $t_f \geq 2t_s/\beta$ by (49). Finally combined with case ii.1, we obtain the desired result. □

Proof of Proposition 4: As in the proof of Lemma 6, we define $\Delta(U,NL)$ as the benefit from brand extension via umbrella branding to a brand when the other brand does not extend, and we define $\Delta(U,U)$ as the benefit from brand extension via umbrella branding to a brand when the other brand extends. Recall from Lemma 6 that $\Delta(U,NL) > 0$ if, and only if, $\lambda_s < \lambda_{sc}^{(1)}$, and that $\Delta(U,U) > 0$ if, and only if, $\lambda_s \in \left(\lambda_{sc}^{(2)}, \lambda_{sc}^{(3)}\right)$.

Now we will show that $\lambda_{sc}^{(2)} < \lambda_{sc}^{(1)} < \lambda_{sc}^{(3)}$ if $v_f - t_f \leq t_s/2\beta$. By (48), (49) and (50), we have $\lambda_{sc}^{(3)} > \lambda_{sc}^{(2)}$, and $\lambda_{sc}^{(3)} > \lambda_{sc}^{(1)}$ when $v_f - t_f \leq t_s/2\beta$. Moreover, by (48), $\lambda_{sc}^{(1)} > \lambda_f$ if $v_f - t_f \leq t_s/2\beta$, and $\lambda_{sc}^{(2)} < \lambda_f$ by (49), which implies that $\lambda_{sc}^{(2)} < \lambda_{sc}^{(1)}$. Next, using above analysis, we prove each part of the proposition separately.

Part I: In this part, $v_f \leq t_f + t_s/2\beta$ so that $\lambda_{sc}^{(1)} < \infty$ by (48), and as is shown above $\lambda_{sc}^{(2)} < \lambda_{sc}^{(1)} < \lambda_{sc}^{(3)}$. Now consider four cases: (a) $\lambda_s \geq \lambda_{sc}^{(3)}$, (b) $\lambda_{sc}^{(3)} > \lambda_s \geq \lambda_{sc}^{(1)}$, (c) $\lambda_{sc}^{(1)} > \lambda_s > \lambda_{sc}^{(2)}$, and (d) $\lambda_s \leq \lambda_{sc}^{(2)}$.

Case I(a): In this case, $\Delta(U,U) < 0$ and $\Delta(U,NL) < 0$ so that it is never optimal for a brand to extend. Consequently, no brand uses umbrella branding.

Case I(b): In this case, $\Delta(U,U) > 0$ and $\Delta(U,NL) < 0$ which implies that it is always optimal for a brand to extend when the other brand also extends and it is optimal to not extend when the other brand does not extend. Thus, there are equilibra:(i) both brands extend, and (ii) no brand extends.

Case I(c): In this case, $\Delta(U,U) > 0$ and $\Delta(U,NL) > 0$ so that each brand extends no matter what the other brand does. Hence, both brands use umbrella branding to extend in this case.

Case I(d): In this case, each brand will use the opposite strategy to the other brand (i.e., it extends if the other brand does not extend, and it does not extend if the other brand extend) since $\Delta(U,U) < 0$ and $\Delta(U,NL) > 0$. Consequently, only one brand use umbrella branding strategy to extend.

Part II: In this part, $v_f > t_f + t_s/2\beta$ so that $\lambda_{sc}^{(1)} = \infty$ by (48), and $\Delta(U,NL) > 0$ by Lemma 6. First consider $\lambda_s \geq \lambda_{sc}^{(3)}$, or $\lambda_s \leq \lambda_{sc}^{(2)}$ (i.e., part II(a) of the proposition). In this case, $\Delta(U,U) \leq 0$ and $\Delta(U,NL) > 0$ so that it is always optimal for a brand to not extend when the other brand extends, and to extend when the other brand does not extend. That is, only one brand extends via umbrella branding. Second consider $\lambda_{sc}^{(2)} < \lambda_s < \lambda_{sc}^{(3)}$ (part II(b) of the proposition). In this case, $\Delta(U,U) > 0$ and $\Delta(U,NL) > 0$, and both brands use umbrella branding to extend. □

Proof of Proposition 5: Comparing (5) and (51), brand A licenses under royalty contract if, and only if, $\lambda_s < v_f + \lambda_f$ and hence part (i) follows.

Next, we prove part (ii) of the proposition and show that the fixed-fee contract (strictly) dominates royalty contract if, and only if: (i) $\lambda_s < \left(\sqrt{2}-1\right)(v_f + \lambda_f)$ when $\lambda_f < 2t_f - v_f$; or (ii) $\lambda_s < v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f$ when $\lambda_f \geq 2t_f - v_f$. By part (i) and Proposition 1, licensing is not optimal under both fixed-fee and royaltycontracts for $\lambda_s \geq v_f + \lambda_f$. Moreover, by Proposition 1, licensing is optimal under fixed-fee contract for $\lambda_s < \check{\lambda}_{sk}^{(1)}$ where $\check{\lambda}_{sk}^{(1)} \in (0, v_f + \lambda_f)$ is given by (53). This by part (i) implies that for $\check{\lambda}_{sk}^{(1)} \leq \lambda_s < v_f + \lambda_f$, brand A licenses under royalty contract while it does not under fixed-fee contract; therefore, royalty contract dominates.

Now, we consider $\lambda_s < \check{\lambda}_{sk}^{(1)}$. From (8) and (51), for $\lambda_f < 2t_f - v_f$, we have $\lambda_s < \check{\lambda}_{sk}^{(1)} = (v_f + \lambda_f)/2$ by (53) and

$$\Pi^A(F) - \Pi^A(R) = \frac{\beta}{8t_f}\left(v_f + \lambda_f - \left(1+\sqrt{2}\right)\lambda_s\right)\left(v_f + \lambda_f + \left(\sqrt{2}-1\right)\lambda_s\right).$$

Thus $\Pi^A(F) - \Pi^A(R) > 0$ and fixed-fee contract dominates for $\lambda_s \leq (v_f + \lambda_f)/\left(1+\sqrt{2}\right)$, and royalty contract dominates for $\lambda_s \in \left((v_f + \lambda_f)/\left(1+\sqrt{2}\right), (v_f + \lambda_f)/2\right)$ when $\lambda_f < 2t_f - v_f$. Now consider $\lambda_f \geq 2t_f - v_f$. In this case, $\lambda_s < \check{\lambda}_{sk}^{(1)} = v_f + \lambda_f - t_f$ by (53). Thus, from (8) and (51), $\Pi^A(F) - \Pi^A(R) > 0$ for $\lambda_s \leq v_f + \lambda_f - 4t_f$, and for $\lambda_s > v_f + \lambda_f - 4t_f$, we have

$$\Pi^A(F) - \Pi^A(R) = \frac{\beta}{8t_f}\left(\lambda_s - v_f - \lambda_f + 2\left(2+\sqrt{2}\right)t_f\right)\left(v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f - \lambda_s\right).$$

Note that $\Pi^A(F) - \Pi^A(R) > 0$ for $\lambda_s \in \left(v_f + \lambda_f - 4t_f, v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f\right)$ and $\Pi^A(F) - \Pi^A(R) < 0$ for $\lambda_s \in \left(v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f, v_f + \lambda_f - t_f\right)$. Then it follows that fixed-fee contract dominates for $\lambda_s < v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f$, and royalty contract dominates for $\lambda_s \in \left[v_f + \lambda_f - 2\left(2-\sqrt{2}\right)t_f, v_f + \lambda_f - t_f\right)$ when $\lambda_f \geq 2t_f - v_f$. □

Proof of Lemma 8: By (31) and (32), we have

$$\Pi^A(F) - \Pi^A(NL) = \frac{(v_s - c - \beta\lambda_s)^2}{4t_s} + \left(v_f + \lambda_f \frac{v_s - c - \beta\lambda_s}{2t_s} - t_f\right)\beta - \frac{(v_s-c)^2}{4t_s}.$$

Note that $\Pi^A(F) - \Pi^A(NL)$ is decreasing in λ_s, $\lim_{\lambda_s \downarrow 0} \Pi^A(F) - \Pi^A(NL) > 0$ and

$$\lim_{\lambda_s \uparrow \frac{v_s - c}{\beta}} \Pi^A(F) - \Pi^A(NL) = (v_f - t_f)\beta - \frac{(v_s-c)^2}{4t_s} < 0,$$

where the last inequality follows from $v_f \le t_f + \frac{(v_s-c)^2}{4\beta t_s}$. Then it follows from that there exists a threshold $\lambda_s \in \left(0, \frac{v_s-c}{\beta}\right)$ such that $\Pi^A(F) > \Pi^A(NL)$ if, and only if, λ_s is less than that threshold. Solving $\Pi^A(F) - \Pi^A(NL) = 0$ for λ_s, it turns out that when $v_f \le t_f + \frac{(v_s-c)^2}{4\beta t_s}$, $\Pi^A(F) > \Pi^A(NL)$ if, and only if,

$$\lambda_s < \lambda_f + \frac{v_s - c}{\beta} - \sqrt{\lambda_f^2 + \left(\frac{v_s-c}{\beta}\right)^2 - 4t_s \frac{v_f - t_f}{\beta}}.$$

Hence, the desired result follows. □

Proof of Lemma 9: By (34), $\theta_s > 0$ if $\lambda_f (v_s - c) > v_f t_s$. Also by (34), for $(v_s - c) < 2t_s$, we have

$$\begin{aligned} \theta_s &= \frac{2t_f t_s (v_s - c) t_f + \beta\lambda_s t_f (\lambda_f (v_s - c) - v_f t_s)}{(t_f t_s + \beta\lambda_f\lambda_s)(4t_f t_s + \beta\lambda_f\lambda_s)} \\ &< \frac{2(2t_f t_s + \beta\lambda_f\lambda_s) t_s t_f - \beta\lambda_s v_f t_s t_f}{(t_f t_s + \beta\lambda_f\lambda_s)(4t_f t_s + \beta\lambda_f\lambda_s)}. \end{aligned} \tag{59}$$

Subtracting numerator of the right hand side of the last inequality above from the denominator, we obtain

$$\beta\lambda_s \left(\beta\lambda_f^2\lambda_s + 3\lambda_f t_f t_s + t_f v_f t_s\right) > 0.$$

This indicates that in (59), the denominator is always greater than the numerator and hence the right hand side of (59) is always less than 1 so that $\theta_s < 1$ for $(v_s - c) < 2t_s$.

By (35), $\theta_f > 0$. Next, we will show that $\theta_f < 0$ if $\beta\lambda_s > v_s - c$. From (35), we have

$$\begin{aligned} \theta_f &= \frac{\lambda_f t_f t_s (v_s - c) + v_f t_s (2t_f t_s + \beta\lambda_f\lambda_s)}{(t_f t_s + \beta\lambda_f\lambda_s)(4t_f t_s + \beta\lambda_f\lambda_s)} \\ &< \frac{\beta\lambda_s\lambda_f t_f t_s + v_f t_s (2t_f t_s + \beta\lambda_f\lambda_s)}{(t_f t_s + \beta\lambda_f\lambda_s)(4t_f t_s + \beta\lambda_f\lambda_s)} \end{aligned} \tag{60}$$

where the inequality follows from $\beta\lambda_s > v_s - c$. Subtracting numerator in the right hand side of the inequality from the denominator, we obtain

$$(2t_f t_s + \beta\lambda_f\lambda_s)(2t_f t_s - v_f t_s + \beta\lambda_f\lambda_s) > 0$$

where the inequlity follows from $\lambda_f (v_s - c) > v_f t_s$ and $\beta\lambda_s > v_s - c$. This indicates that the right hand side of inequality in (60) is always less than 1 and hence $\theta_f < 1$. □